1

(a)

$$\text{P(Score = 5)} = \frac{4}{5} \times \frac{\pi(10)^2}{\pi(30)^2} = \frac{4}{45}$$

$$\text{P(Score = 3)} = \frac{4}{5} \times \frac{\pi(20)^2 - \pi(10)^2}{\pi(30)^2} = \frac{4}{15}$$

$$\text{P(Score = 1)} = \frac{4}{5} \times \frac{\pi(30)^2 - \pi(20)^2}{\pi(30)^2} = \frac{4}{9} \quad \text{(Shown)}$$

(b)

$$\text{P}(X = 0) = \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) = \frac{1}{25}$$

$$\text{P}(X = 1) = \left(\frac{1}{5}\right)\left(\frac{4}{9}\right)(2) + \left(\frac{4}{9}\right)^2 = \frac{152}{405}$$

$$\text{P}(X = 3) = \left(\frac{1}{5} + \frac{4}{9}\right)\left(\frac{4}{15}\right)(2) + \left(\frac{4}{15}\right)^2 = \frac{56}{135}$$

$$\text{P}(X = 5) = \left(\frac{1}{5} + \frac{4}{9} + \frac{4}{15}\right)\left(\frac{4}{45}\right)(2) + \left(\frac{4}{45}\right)^2 = \frac{344}{2025}$$

x	0	1	3	5
$\text{P}(X = x)$	$\frac{1}{25}$	$\frac{152}{405}$	$\frac{56}{135}$	$\frac{344}{2025}$

(c)

$$\text{E}(X) = 0\left(\frac{1}{25}\right) + 1\left(\frac{152}{405}\right) + 3\left(\frac{56}{135}\right) + 5\left(\frac{344}{2025}\right)$$

$$= \frac{200}{81}$$

2

(a) $k+2k+3k+4k+5k+6k=1$

$21k=1$

$\therefore k=\dfrac{1}{21}$ (Shown)

(b) $\mathrm{E}(X)=1(k)+2(2k)+3(3k)+4(4k)+5(5k)+6(6k)$

$\mathrm{E}(X)=91k=91\left(\dfrac{1}{21}\right)=\dfrac{13}{3}$

3

(i)

	1	2	3	4	5
1	X	3	4	5	6
2	3	X	5	6	7
3	4	5	X	7	8
4	5	6	7	X	9
5	6	7	8	9	X

$\mathrm{P}(X=3)=2\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{10}$ $\quad$ $\mathrm{P}(X=7)=4\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{5}$

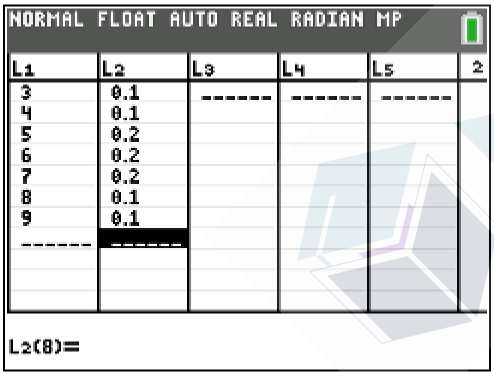
$\mathrm{P}(X=4)=2\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{10}$ $\quad$ $\mathrm{P}(X=8)=2\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{10}$

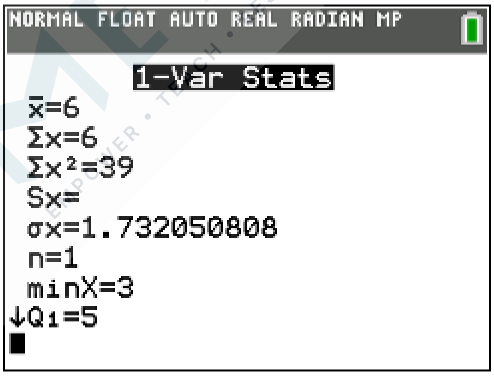
$\mathrm{P}(X=5)=4\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{5}$ $\quad$ $\mathrm{P}(X=9)=2\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{10}$

$\mathrm{P}(X=6)=4\times\frac{1}{5}\times\frac{1}{4}=\frac{1}{5}$

x	3	4	5	6	7	8	9
P(X=x)	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

(ii)





$\mathrm{E}(X)=6$

$\mathrm{Var}(X)=1.7320^2$

$\quad\quad=2.9998\approx 3.00$ (3 s.f.)

4

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(ii) $\mathrm{E}(X)$

$$=(1)\left(\frac{1}{36}\right)+(2)\left(\frac{3}{36}\right)+(3)\left(\frac{5}{36}\right)+(4)\left(\frac{7}{36}\right)$$

$$+(5)\left(\frac{9}{36}\right)+(6)\left(\frac{11}{36}\right)$$

$$=\frac{161}{36}\approx 4.472$$

$$\mathrm{E}\left(X^2\right)=(1)^2\left(\frac{1}{36}\right)+(2)^2\left(\frac{3}{36}\right)+(3)^2\left(\frac{5}{36}\right)+(4)^2\left(\frac{7}{36}\right)$$

$$+(5)^2\left(\frac{9}{36}\right)+(6)^2\left(\frac{11}{36}\right)$$

$$=\frac{791}{36}$$

$$\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\mathrm{E}(X)^2$$

$$=\frac{791}{36}-\left(\frac{161}{36}\right)^2$$

$$=\frac{2555}{1296}\approx 1.97$$

5

(a) $\sum P(X = x) = 1$

$3k + 4k + 5k + 5k + 4k + 3k = 1$

$24k = 1 \Rightarrow k = \frac{1}{24}$ (shown)

(b) By symmetry, $E(X) = 5.5$

$$E(X^2) = \frac{3}{24}(3^2) + \frac{4}{24}(4^2) + \frac{5}{24}(5^2) + \frac{5}{24}(6^2) + \frac{4}{24}(7^2) + \frac{3}{24}(8^2)$$

$$= \frac{98}{3}$$

$$\text{Var}(X) = \frac{98}{3} - \left(\frac{11}{2}\right)^2 = \frac{29}{12}$$

(c) Since n = 50 is large, by Central Limit Theorem,

$\overline{X} \sim N\left(5.5, \frac{29/12}{50}\right)$ approx ie $\overline{X} \sim N\left(5.5, \frac{29}{600}\right)$ approx

$P(\overline{X} < 4.9) = 0.00317$ (3 s.f)

(d) $E(Y^2) = E\left((mX - 1)^2\right)$

$= E(m^2X^2 - 2mX + 1)$

$= m^2E(X^2) - 2mE(X) + 1$

$= \frac{98}{3}m^2 - 2m(5.5) + 1$

$= \frac{98}{3}m^2 - 11m + 1$

(e) $P(Y \geq a) \geq 0.5$

$P(3X - 1 \geq a) \geq 0.5$

$P\left(X \geq \frac{a+1}{3}\right) \geq 0.5$

Since

$P(X \geq 5) = 0.70833 > 0.5$

$P(X \geq 6) = 0.5$

$\therefore \frac{a+1}{3} \leq 6 \Rightarrow a \leq 17$

Largest $a = 17$

6

(a) $k = \frac{1}{7}$

(b) Let G be absolute difference of two scores.
Probability Distribution of G:

g	0	1	3	4
$\mathrm{P}(G=g)$	$\left(\frac{1}{7}\right)\left(\frac{1}{7}\right)+\left(\frac{2}{7}\right)\left(\frac{2}{7}\right)+\left(\frac{4}{7}\right)\left(\frac{4}{7}\right)=\frac{21}{49}$	$2\left(\frac{2}{7}\right)\left(\frac{4}{7}\right)=\frac{16}{49}$	$2\left(\frac{1}{7}\right)\left(\frac{2}{7}\right)=\frac{4}{49}$	$2\left(\frac{1}{7}\right)\left(\frac{4}{7}\right)=\frac{8}{49}$

$$\mathrm{E}(G)=1\left(\frac{16}{49}\right)+3\left(\frac{4}{49}\right)+4\left(\frac{8}{49}\right)$$
$$=\frac{60}{49}$$
$$\mathrm{E}(2G-m)>0$$
$$2\mathrm{E}(\mathrm{G})-m>0$$
$$2\left(\frac{60}{49}\right)-m>0$$
$$\therefore 0<m<\frac{120}{49}=2.45$$

(c) Tim's winnings is based on $\mathrm{E}(G)$, which is the **long term** average score. He may still lose for some of the 2 games, but in the long run, he makes a profit.

7

Since $\mathrm{P}(T=1)=\mathrm{P}(T=4)=\frac{1}{6}$, then $\mathrm{P}(T=2)+\mathrm{P}(T=3)=\frac{4}{6}$.

If $\mathrm{P}(T=2)=\mathrm{P}(T=3)=\frac{2}{6}$, then the modes of T are 2 and 3, which contradicts the question.

Hence, $\mathrm{P}(T=2)=\frac{3}{6}=\frac{1}{2}$ and $\mathrm{P}(T=3)=\frac{1}{6}$.

t	1	2	3	4
$\mathrm{P}(T=t)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$

(i) $\mathrm{P}(Y=0)=\mathrm{P}(1\text{ or }3)+\mathrm{P}(2)\times\mathrm{P}(1\text{ or }3)+\mathrm{P}(2)\times\mathrm{P}(2)\times\mathrm{P}(1\text{ or }3)+\left[\mathrm{P}(2)\right]^3\times\mathrm{P}(1\text{ or }3)+\ldots$

$$=\frac{2}{6}+\left(\frac{1}{2}\right)\left(\frac{2}{6}\right)+\left(\frac{1}{2}\right)^2\left(\frac{2}{6}\right)+\left(\frac{1}{2}\right)^3\left(\frac{2}{6}\right)+\ldots$$

$$=\frac{\frac{2}{6}}{1-\frac{1}{2}}$$

$$=\frac{2}{3}\quad\text{(shown)}$$

(ii)

y	0	1	2	3	…
$\mathrm{P}(Y=y)$	$\frac{2}{3}$	$\frac{1}{6}$	$\left(\frac{1}{2}\right)\frac{1}{6}$	$\left(\frac{1}{2}\right)^2\frac{1}{6}$	

$$\mathrm{E}\left(Y^2\right)=0+1^2\left(\frac{1}{6}\right)+2^2\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)+3^2\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right)+\ldots$$

$$=\frac{1}{6}\left(1^2+2^2\left(\frac{1}{2}\right)+3^2\left(\frac{1}{2}\right)^2+\ldots\right)$$

$$=\frac{1}{6}\times\frac{1+\frac{1}{2}}{\left(1-\frac{1}{2}\right)^3}=2$$

$$\mathrm{Var}(Y)=\mathrm{E}\left(Y^2\right)-\left(\mathrm{E}(Y)\right)^2=2-\left(\frac{2}{3}\right)^2=\frac{14}{9}$$

8

Tables of outcomes

	1	2	3	4
1	3	3	4	5
2	3	6	5	6
3	4	5	9	7
4	5	6	7	12

$$\mathrm{P}(\text{spin}=1)=\frac{144}{360}=\frac{2}{5}=\frac{4}{10}$$

$$\mathrm{P}(\text{spin}=2)=\frac{108}{360}=\frac{3}{10}$$

$$\mathrm{P}(\text{spin}=3)=\frac{72}{360}=\frac{1}{5}=\frac{2}{10}$$

$$\mathrm{P}(\text{spin}=4)=\frac{36}{360}=\frac{1}{10}$$

(a) $\mathrm{P}(X=6)$

$=\mathrm{P}(\text{spin}_1=2,\text{spin}_2=2)+\mathrm{P}(\text{spin}_1=2,\text{spin}_2=4)+\mathrm{P}(\text{spin}_1=4,\text{spin}_2=2)$

$=\left[\left(\frac{3}{10}\right)\left(\frac{3}{10}\right)\right]+\left[\left(\frac{3}{10}\right)\left(\frac{1}{10}\right)\right]+\left[\left(\frac{1}{10}\right)\left(\frac{3}{10}\right)\right]$

$=0.15$

(b) $P(X=3)$

$= P(\text{spin}_1 = 1, \text{spin}_2 = 1) + P(\text{spin}_1 = 1, \text{spin}_2 = 2) + P(\text{spin}_1 = 2, \text{spin}_2 = 1)$

$$= \left[\left(\frac{4}{10}\right)\left(\frac{4}{10}\right)\right] + \left[\left(\frac{4}{10}\right)\left(\frac{3}{10}\right)\right] + \left[\left(\frac{3}{10}\right)\left(\frac{4}{10}\right)\right]$$

$= 0.4$

$P(X=4)$

$= P(\text{spin}_1 = 1, \text{spin}_2 = 3) + P(\text{spin}_1 = 3, \text{spin}_2 = 1)$

$$= \left[\left(\frac{4}{10}\right)\left(\frac{2}{10}\right)\right] + \left[\left(\frac{2}{10}\right)\left(\frac{4}{10}\right)\right]$$

$= 0.16$

$P(X=5)$

$= P(\text{spin}_1 = 1, \text{spin}_2 = 4) + P(\text{spin}_1 = 4, \text{spin}_2 = 1) + P(\text{spin}_1 = 2, \text{spin}_2 = 3) + P(\text{spin}_1 = 3, \text{spin}_2 = 2)$

$$= \left[\left(\frac{4}{10}\right)\left(\frac{1}{10}\right)\right] + \left[\left(\frac{1}{10}\right)\left(\frac{4}{10}\right)\right] + \left[\left(\frac{3}{10}\right)\left(\frac{2}{10}\right)\right] + \left[\left(\frac{2}{10}\right)\left(\frac{3}{10}\right)\right]$$

$= 0.2$

$P(X=7)$

$= P(\text{spin}_1 = 3, \text{spin}_2 = 4) + P(\text{spin}_1 = 4, \text{spin}_2 = 3)$

$$= \left[\left(\frac{2}{10}\right)\left(\frac{1}{10}\right)\right] + \left[\left(\frac{1}{10}\right)\left(\frac{2}{10}\right)\right]$$

$= 0.04$

$P(X=9)$

$= P(\text{spin}_1 = 3, \text{spin}_2 = 3)$

$$= \left[\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)\right]$$

$= 0.04$

$\mathrm{P}(X=12)$

$=\mathrm{P}(\text{spin}_1=4,\text{spin}_2=4)$

$=0.01$

Alternative presentation format (Probability Distribution Table)

x	3	4	5	6	7	9	12
$\mathrm{P}(X=x)$	$\frac{40}{100}$ $=\frac{2}{5}$ $=0.4$	$\frac{16}{100}$ $=\frac{4}{25}$ $=0.16$	$\frac{20}{100}$ $=\frac{1}{5}$ $=0.2$	$\frac{15}{100}$ $=\frac{3}{20}$ $=0.15$	$\frac{4}{100}$ $=\frac{1}{25}$ $=0.04$	$\frac{4}{100}$ $=\frac{1}{25}$ $=0.04$	$\frac{1}{100}$ $=0.01$

(c) $\mathrm{P}(\text{Score} < 10 \mid \text{Customer wins a prize})$

$=\mathrm{P}(X<10 \mid X>6)$

$=\dfrac{\mathrm{P}(X<10 \cap X>6)}{\mathrm{P}(X>6)}$

$=\dfrac{\mathrm{P}(X=7)+\mathrm{P}(X=9)}{\mathrm{P}(X=7)+\mathrm{P}(X=9)+\mathrm{P}(X=12)}$

$=\dfrac{(0.04)+(0.04)}{(0.04)+(0.04)+(0.01)}$

$=\dfrac{8}{9}$ or 0.889 (3 s.f.)

9

(a) Area of target board = $\pi\left(5^2\right)=25\pi$

Area of region with score 50 $=\pi\left(1^2\right)=\pi$

Probability of dart hitting region with score 50 $=\dfrac{\pi}{25\pi}=\dfrac{1}{25}$

Area of region with score $25=\pi\left(3^2\right)-\pi=8\pi$

Probability of dart hitting region with score $25=\dfrac{8\pi}{25\pi}=\dfrac{8}{25}$

$$\mathrm{P}(S=75)=\frac{8}{25}\times\frac{1}{25}\times 2!$$
$$=\frac{16}{625} \quad \text{(shown)}$$

(b) Area of region with score 0 $=25\pi-9\pi=16\pi$

Probability of dart hitting region with score 0 $=\dfrac{16\pi}{25\pi}=\dfrac{16}{25}$

s	0	25	50	75	100
$\mathrm{P}(S=s)$	$\frac{16}{25}\times\frac{16}{25}$ $=\frac{256}{625}$	$\frac{16}{25}\times\frac{8}{25}\times 2!$ $=\frac{256}{625}$	$\frac{16}{25}\times\frac{1}{25}\times 2!$ $+\frac{8}{25}\times\frac{8}{25}$ $=\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{25}\times\frac{1}{25}$ $=\frac{1}{625}$

(c) From G.C.,
$\mathrm{E}(S)=20$ and $\mathrm{Var}(S)=400$.

10

(a)

x	1	2	3	4
$P(X=x)$	0.42	0.42	0.11	0.05

Table of outcomes

X_1 / X_2	1	2	3	4
1	2	3	4	4
2	3	4	5	4
3	4	5	6	4
4	5	6	7	4

y	2	3	4	5	6	7
$P(Y=y)$	0.42^2 $= 0.1764$	$2(0.42)$ $= 0.3528$	0.42^2 $+2(0.42)(0.11)$ $+(0.05)(1)$ $= 0.3528$ 0.3188	$(0.42)(0.05)$ $+2(0.42)(0.11)$ $= 0.1134$	$(0.42)(0.05)$ $+0.11^2$ $= 0.0331$	$(0.11)(0.05)$ $= 0.0055$

(b) $E(Y) = 3.4905$

$$\text{Var}(Y) = 1.045997012^2$$
$$= 1.0941$$
$$= 1.09 \text{ (3 s.f.)}$$

(c) Let W be the winning from a game.

w	0	20	50
$P(W=w)$	0.848	0.1465	0.0055

$$E(W) = 20(0.1465) + 50(0.0055) = 3.205$$

Expected loss for a game = 5 – 3.205 = 1.795
Expected loss for 40 games = 1.796 x 40 = $71.80

11

(a)

$3a+b=1 \quad \text{-- } (*)$

$\mathrm{E}(S)=6a+4b=2.56 \quad \text{-- } (*)$

$3a+2b=1.28$

Subtracting one eqn from the other,

$$b=1.28-1=0.28=\frac{7}{25}$$

(b)(i)

From part (a), $a=\frac{6}{25}$

$$\mathrm{P}\left(S_1+S_2\geq 6 \middle| \text{one of the scores is } 3\right)$$

$$=\frac{\mathrm{P}\left((S_1+S_2=3)\ \cap(S_1=3 \text{ OR } S_2=3)\right)}{\mathrm{P}\left(S_1=3 \text{ OR } S_2=3\right)}$$

$$=\frac{\mathrm{P}\left(S_1=3,S_2=4\right)+\mathrm{P}\left(S_1=4,S_2=3\right)+\mathrm{P}\left(S_1=3,S_2=3\right)}{\mathrm{P}\left(S_1=3,S_2\neq 3\right)+\mathrm{P}\left(S_1\neq 3,S_2=3\right)+\mathrm{P}\left(S_1=3,S_2=3\right)}$$

$$=\frac{2\left(\frac{6}{25}\right)\left(\frac{7}{25}\right)+\left(\frac{6}{25}\right)^2}{2\left(\frac{6}{25}\right)\left(\frac{19}{25}\right)+\left(\frac{6}{25}\right)^2}=\frac{120}{264}=\frac{5}{11}=0.455 \text{ (3 s.f.)}$$

(b)(ii)

From part (a), $a=\frac{6}{25}$

$$\mathrm{Var}\left(2S-\mathrm{E}(Y)\right)$$

$$=\mathrm{Var}(2S) \quad \left(\because \mathrm{E}(Y) \text{ is a constant and } \mathrm{Var}(\text{constant})=0\right)$$

$$=4\mathrm{Var}(S)$$

$$=4\left(\mathrm{E}\left(S^2\right)-2.56^2\right)$$

$$=4\left(a+4a+9a+16b-2.56^2\right)$$

$$=4(14a+16b-6.5536)$$

$$=5.1456$$

(b)(iii)

$$\begin{aligned}\mathrm{E}\left(Y-\mathrm{E}(Y)\right)&=\mathrm{E}(Y)-\mathrm{E}\left(\mathrm{E}(Y)\right)\\&=\mathrm{E}(Y)-\mathrm{E}(Y)\\&=0\end{aligned}$$

12

(i) The possible values of X are 2, 3, 4 and 5

$$\mathrm{P}(X=2)=\mathrm{P}(RR)=\frac{5}{8}\times\frac{4}{7}=\frac{5}{14}$$

$$\mathrm{P}(X=3)=\mathrm{P}(RBR \text{ or } BRR)=\frac{5}{8}\times\frac{3}{7}\times 2!\times\frac{4}{6}=\frac{5}{14}$$

$\mathrm{P}(X=4)$

$=\mathrm{P}(RBBR,$ the first 3 balls can be in any order but last one must be $R)$

$$=\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{3!}{2!}\times\frac{4}{5}=\frac{3}{14}$$

$\mathrm{P}(X=5)$

$=\mathrm{P}(RBBBR,$ the first 4 balls can be in any order but last one must be $R)$

$$=\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\times\frac{4!}{3!}\times\frac{4}{4}=\frac{1}{14}$$

(ii)

$$\mathrm{E}(X)=\sum_{\text{all } x} x\ \mathrm{P}(X=x)$$

$$=2\times\frac{5}{14}+3\times\frac{5}{14}+4\times\frac{3}{14}+5\times\frac{1}{14}$$

$$=3$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$$

$$=2^2\times\frac{5}{14}+3^2\times\frac{5}{14}+4^2\times\frac{3}{14}+5^2\times\frac{1}{14}-[3]^2$$

$$=\frac{69}{7}-9=\frac{6}{7}$$

(iii) Expected profit = Expected Loss

$\$y=\$2\times\mathrm{P}(X=2)+\$3\times\mathrm{P}(X=3)+\$4\times\mathrm{P}(X=4)+\$5\times\mathrm{P}(X=5)$

$$y=2\times\frac{5}{14}+3\times\frac{5}{14}+4\times\frac{3}{14}+5\times\frac{1}{14}$$

$$y=3$$

13

(a) Group the 2 red discs and the blue disc as one unit.
Together with the remaining 6 green discs, there are 7 units.

Required no. of arrangements

$$=\frac{7!}{6!}\times\frac{3!}{2!}$$
$$=21$$

(b)(i)

$$P(R=0)=\frac{{}^{2}C_{0}\times{}^{7}C_{3}}{{}^{9}C_{3}}=\frac{5}{12} \quad \text{or} \quad \frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}=\frac{5}{12}$$

$$P(R=1)=\frac{{}^{2}C_{1}\times{}^{7}C_{2}}{{}^{9}C_{3}}=\frac{1}{2} \quad \text{or} \quad \frac{2}{9}\times\frac{7}{8}\times\frac{6}{7}\times 3=\frac{1}{2}$$

$$P(R=2)=\frac{{}^{2}C_{2}\times{}^{7}C_{1}}{{}^{9}C_{3}}=\frac{1}{12} \quad \text{or} \quad \frac{2}{9}\times\frac{1}{8}\times\frac{7}{7}\times 3=\frac{1}{12}$$

$$\text{or} \quad 1-\frac{5}{12}-\frac{1}{2}=\frac{1}{12}$$

Probability distribution of R:

r	0	1	2
$P(R=r)$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$

(b)(ii)

r	0	1	2
$P(R=r)$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$
Change in points	$0(9)+3(-3)$ $=-9$	$1(9)+2(-3)$ $=3$	$2(9)+1(-3)$ $=15$

Expected change in points

$$=(-9)P(R=0)+(3)P(R=1)+(15)P(R=2)$$
$$=(-9)\left(\frac{5}{12}\right)+(3)\left(\frac{1}{2}\right)+(15)\left(\frac{1}{12}\right)$$
$$=-1$$

14

(a)

$$\frac{1}{3r+1}+\left(\frac{3r}{3r+1}\right)^2\times\frac{1}{3r+1}+\left(\frac{3r}{3r+1}\right)^4\times\frac{1}{3r+1}+\ldots$$

$$=\frac{\frac{1}{3r+1}}{1-\left(\frac{3r}{3r+1}\right)^2}=\frac{3r+1}{6r+1}$$

(b)(i)

$$\mathrm{P}(T=0)=\frac{r}{3r+1}\times\frac{r-1}{3r}+\frac{2r}{3r+1}\times\frac{2r-1}{3r}=\frac{5r-3}{3(3r+1)}$$

$$\mathrm{P}(T=2)=\frac{2r}{3r+1}\times\frac{1}{3r}\times 2!=\frac{4}{3(3r+1)}$$

$$\mathrm{P}(T=3)=\frac{2r}{3r+1}\times\frac{r}{3r}\times 2!=\frac{4r}{3(3r+1)}$$

$$\mathrm{P}(T=5)=\frac{r}{3r+1}\times\frac{1}{3r}\times 2!=\frac{2}{3(3r+1)}$$

t	0	2	3	5
$\mathrm{P}(T=t)$	$\frac{5r-3}{3(3r+1)}$	$\frac{4}{3(3r+1)}$	$\frac{4r}{3(3r+1)}$	$\frac{2}{3(3r+1)}$

(b)(ii)

$$\mathrm{E}(T)=\sum t\mathrm{P}(T=t)$$

$$=0\times\frac{10r-4}{3(3r+1)}+2\times\frac{4}{3(3r+1)}+3\times\frac{4r}{3(3r+1)}+5\times\frac{2}{3(3r+1)}$$

$$=\frac{8+12r+10}{3(3r+1)}=\frac{2(2r+3)}{3r+1}$$

(c)

x	-0.25	0.10	0.15
$\mathrm{P}(X=x)$	$\frac{r}{3r+1}$	$\frac{2r}{3r+1}$	$\frac{1}{3r+1}$

$$\mathrm{P}(X_1>X_2)=\mathrm{P}(X_1=0.15, X_2=-0.25)$$
$$+\mathrm{P}(X_1=0.15, X_2=0.10)$$
$$+\mathrm{P}(X_1=0.10, X_2=-0.25)$$

$$\frac{2r^2+3r}{(3r+1)^2}=\frac{27}{112}$$

$$19r^2-174r+27=0$$

$$r=\frac{3}{19}\approx 0.15789 \text{ (rejected)} \qquad \therefore r=9$$

$$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$$

$$=-0.25\times\frac{9}{28}+0.10\times\frac{18}{28}+0.15\times\frac{1}{28}$$

$$=-0.010714\approx -0.0107$$

Since $\mathrm{E}(x)<0$, the game is not fair for a player, as he/she will be expected to lose 1.07 cents every time he/she plays.

15

(a)

$$E(S)=0.55(1.2\times10)+0.45\left(\frac{1}{1.2}\times10\right)$$
$$=10.35$$

$$E(S^2)=0.55(1.2\times10)^2+0.45\left(\frac{1}{1.2}\times10\right)^2$$
$$=110.45$$

$$\mathrm{Var}(S)=E(S^2)-[E(S)]^2$$
$$=110.45-10.35^2$$
$$=3.3275$$

(b)(i)

$$S_0(u^4)\left(\frac{1}{u}\right)^2=u^2S_0$$

(b)(i)

$$\text{The required probability}=\binom{6}{4}p^4(1-p)^2$$
$$=15p^4(1-p)^2$$

(b)(ii) If the stock price goes up 3 times and goes down 3 times, the end price is exactly S_0, so there must be at least 4 "up"s.

Let X denote the number of "Up"s in the 6 periods.

$X \sim B(6,p)$

$$P(X\ge4)=\binom{6}{4}p^4(1-p)^2+\binom{6}{5}p^5(1-p)^1+\binom{6}{6}p^6$$
$$=15p^4(1-p)^2+6p^5(1-p)+p^6$$

(c)(i) $10\times1.2^2-10=4.4$

$X \sim B(6,0.55)$

$P(X\le3)=0.55848$ and $P(X=4)=0.27795$

Number of times for the stock price to rise	3 or below	4	5	6
v	0	4.4	10.736	19.85984
$P(V=v)$	0.55848	0.27795	0.13589	0.02768

(c)(ii) $\mathrm{E}(V) = 4.4 \times 0.27795 + 10.736 \times 0.13589 + 19.85984 \times 0.02768$

$= 3.2316154$

The expected return is

$= 4000(3.2316154)$

$= 12926.46$

$= 12926$ (nearest dollar)

$$\begin{aligned}\mathrm{E}(V) = &\left(10 \times 1.2^2 - 10\right)\mathrm{P}(X = 4) \\ &+ \left(10 \times 1.2^4 - 10\right)\mathrm{P}(X = 5) \\ &+ \left(10 \times 1.2^6 - 10\right)\mathrm{P}(X = 6) \\ = &\ 3.23159461\end{aligned}$$

The expected return is

$= 4000(3.2316154)$

$= 12926.46$

$= 12926$ (nearest dollar)

(c)(iii) Acceptable answer 1:

Good investment as the expectation is higher than 10000.

Acceptable answer 2:

By GC, $\sigma_V = 4.6387546$

The return $4000V$ has a standard deviation

$4.6387546 \times 4000 = 18555$ (nearest dollar)

Risky investment as the standard deviation (or variance) is large (in comparison / relative to the expectation).

Acceptable answer 3:

Calculate the expectation of using \$10000 to buy the stock directly.

$$10000 \sum_{r=0}^{6} \left(1.2^{2r-6} - 1\right)\mathrm{P}(X = r) = 12293 \text{ (nearest dollar)}$$

Good investment as the expectation is higher than buying the stock directly.