

1

Let  $z = x + yi$

$$(x + yi + i)^* = 2i(x + yi) + 4$$

$$[x + i(y + 1)]^* = 2i(x + yi) + 4$$

$$x - i(y + 1) = 2ix - 2y + 4$$

Comparing real and imaginary part

$$x = 4 - 2y \text{ --- (1)}$$

$$-(y + 1) = 2x \text{ --- (2)}$$

solving:

$$-y - 1 = 2(4 - 2y)$$

$$3y = 9$$

$$y = 3$$

Solving,  $x = -2$ ,  $y = 3$



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**2**

$$4z + 5iw = 7 \quad (1)$$

$$(1-i)z + 8w = 30 \quad (2)$$

Multiply (2) by  $2(1+i)$ :  $4z + 16(1+i)w = 60(1+i)$  (3)

$$(3) - (1): \quad (16 + 11i)w = 53 + 60i$$

$$w = 4 + i$$

From (1):  $4z + 5i(4+i) = 7$

$$z = 3 - 5i$$



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3

$$w + z^* = -2 + 4i \text{ ----(1)}$$

$$z + 2 = 3iw \text{ ----(2)}$$

From (1):  $w = -2 + 4i - z^*$

Substituting into (2):  $z + 2 = 3i(-2 + 4i - z^*)$

$$z + 3iz^* = -14 - 6i$$

Let  $z = a + ib$ ,

$$a + ib + 3ia + 3b = -14 - 6i$$

$$(a + 3b) + (3a + b)i = -14 - 6i$$

Comparing real and imaginary parts:

$$a + 3b = -14 \text{ ---(3)}$$

$$3a + b = -6 \text{ ---(4)}$$

Solving simultaneously with (4) - 3(3), we have

$$-8b = 36 \Rightarrow b = -4.5$$

Hence  $a = -0.5$ , and  $z = -0.5 - 4.5i$

$$w = -2 + 4i - z^*: w = -2 + 4i + 0.5 - 4.5i = -1.5 - 0.5i$$



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$$z^* + iw = -6i \text{ ---(1)}$$

$$(3 - i)z + 2w = -9 + 5i \text{ ---(2)}$$

From equation (1):

$$w = \frac{-6i - z^*}{i} = -6 + iz^*$$

Sub  $w$  into (2):

$$(3 - i)z + 2(-6 + iz^*) = -9 + 5i$$

$$3z - iz - 12 + 2iz^* = -9 + 5i$$

$$3z - iz + 2iz^* = 3 + 5i$$

Let  $z = a + bi$

$$3(a + bi) - i(a + bi) + 2i(a - bi) = 3 + 5i$$

$$3a + 3bi - ia + b + 2ia + 2b = 3 + 5i$$

$$3a + 3b + 3bi + ai = 3 + 5i$$

Comparing the real and imaginary parts,

$$3a + 3b = 3 \text{ and } 3b + a = 5$$

$$\therefore a = -1, b = 2$$

$$z = -1 + 2i$$

$$w = -6 + i(-1 - 2i) = -4 - i$$



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(i) Since the coefficients of the polynomial are all real, as  $z = ki$  is a root, its conjugate  $z = -ki$  is also a root.

$$\begin{aligned}x^4 - 4x^3 + ax^2 - 16x + b &= (x - ki)(x + ki)(x^2 + cx + d) \\ &= (x^2 + k^2)(x^2 + cx + d)\end{aligned}$$

Comparing constant,  $b = dk^2 \Rightarrow d = \frac{b}{k^2} \dots (1)$

Comparing coefficient of  $x$ ,  $-16 = ck^2 \dots (2)$

Comparing coefficient of  $x^2$ ,  $a = d + k^2 \dots (3)$

Comparing coefficient of  $x^3$ ,  $-4 = c \dots (4)$

Substitute (4) into (2),

$$-16 = -4k^2$$

$$k^2 = 4$$

$$k = \pm 2$$

The two roots are  $2i$  and  $-2i$ .

Substitute (1) and  $k^2 = 4$  into (3),

$$a = \frac{b}{4} + 4$$

$$4a = b + 16$$

$$16 - 4a + b = 0 \quad (\text{shown})$$

(ii) Sub  $(0, 20)$  into  $f(x) = x^4 - 4x^3 + ax^2 - 16x + b$ ,  $b = 20$

$$16 - 4a + 20 = 0$$

$$a = 9$$

$$\begin{aligned}\therefore f(x) &= x^4 - 4x^3 + 9x^2 - 16x + 20 \\ &= (x - 2i)(x + 2i)(x^2 + cx + d) \\ &= (x^2 + 4)(x^2 - 4x + 5)\end{aligned}$$

6

$$\begin{aligned} & \frac{2 - i \sin 2\alpha}{1 + 2i \sin 2\alpha} \times \frac{1 - 2i \sin 2\alpha}{1 - 2i \sin 2\alpha} \\ &= \frac{(2 - i \sin 2\alpha)(1 - 2i \sin 2\alpha)}{1 + 4 \sin^2 2\alpha} \\ &= \frac{2 - 2 \sin^2 2\alpha - 5i \sin 2\alpha}{1 + 4 \sin^2 2\alpha} \end{aligned}$$

Since the expression is real,

$$\frac{-5i \sin 2\alpha}{1 + 4 \sin^2 2\alpha} = 0$$

$$-5i \sin 2\alpha = 0$$

$$\sin 2\alpha = 0$$

$$2\alpha = k\pi, \quad k \in \mathbb{Z}$$

$$\alpha = \frac{k\pi}{2}$$

$$\therefore \left\{ \alpha \in \mathbb{R} \mid \alpha = \frac{k\pi}{2}, k \in \mathbb{Z} \right\}$$



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$$\begin{aligned} \text{(i)} \quad & z^4 - kz^3 + k^3z - k^4 = 0 \\ & (z-k)(z+k) = z^2 - k^2 \\ & z^4 - kz^3 + k^3z - k^4 = (z^2 - k^2)(z^2 + cz + d) \end{aligned}$$

Comparing coefficients of

$$\begin{aligned} z^0 : -k^4 &= -k^2d \Rightarrow d = k^2 \\ z^1 : k^3 &= -k^2c \Rightarrow c = -k \\ \therefore z^4 - kz^3 + k^3z - k^4 &= (z^2 - k^2)(z^2 - kz + k^2) \end{aligned}$$

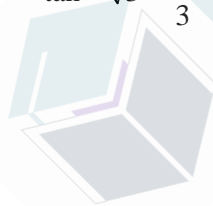
For  $z^2 - kz + k^2$ ,

$$z = \frac{k \pm \sqrt{k^2 - 4k^2}}{2} = \frac{k}{2} \pm \frac{\sqrt{3}k}{2}i$$

The other two roots of the equation are  $z_3 = \frac{k}{2} + \frac{\sqrt{3}k}{2}i$  (shown) and  $z_4 = \frac{k}{2} - \frac{\sqrt{3}k}{2}i$  (as  $\text{Im}(z_3) > 0$  and  $\text{Im}(z_4) < 0$ ).

$$\text{(ii)} \quad |z_3| = \sqrt{\left(\frac{k}{2}\right)^2 + \left(\frac{\sqrt{3}k}{2}\right)^2} = k$$

$$\arg z_3 = \tan^{-1} \frac{(\sqrt{3}k/2)}{(k/2)} = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$



8

- (i) Since  $x = -2 + i$  is a root of  $x^3 + ax^2 - 7x + b = 0$ , it satisfies the equation.

$$(-2+i)^3 + a(-2+i)^2 - 7(-2+i) + b = 0$$

$$12 + 4i + a(3 - 4i) + b = 0$$

$$12 + 3a + b + (4 - 4a)i = 0$$

By comparing real and imaginary parts,

$$12 + 3a + b = 0 \Rightarrow 12 = -3a - b$$

$$4 - 4a = 0 \Rightarrow 4 = 4a$$

$$\Rightarrow a = 1$$

$$\therefore b = -15$$

**Alternative method (remove from examiner report)**

Since all the coefficients of the equation are real,  $x = -2 + i$  is a root means that  $x = -2 - i$  is also a root.

$$\begin{aligned} x^3 + ax^2 - 7x + b &= [x - (-2 + i)][x - (-2 - i)][x + C] \\ &= [(x + 2) - i][(x + 2) + i][x + C] \\ &= [(x + 2)^2 - i^2][x + C] \\ &= (x^2 + 4x + 5)(x + C) \end{aligned}$$

By comparing:

Constant term:  $b = 5C$

Coefficient of  $x$ :  $-7 = 4C + 5 \Rightarrow C = -3$

Coefficient of  $x^2$ :  $a = C + 4$

So,  $b = -15$ ,  $a = 1$

- (ii)  $x^3 + x^2 - 7x - 15 = 0$

Since all the coefficients of the equation are real,  $x = -2 + i$  is a root means that  $x = -2 - i$  is also a root.

$$\begin{aligned} \text{Quadratic factor: } & [x - (-2 + i)][x - (-2 - i)] \\ &= [(x + 2) - i][(x + 2) + i] \\ &= (x + 2)^2 - i^2 \\ &= x^2 + 4x + 5 \end{aligned}$$

$$x^3 + x^2 - 7x - 15 = 0$$

$$(x^2 + 4x + 5)(x + c) = 0$$

By comparing coefficients or long-division,

$$5c = -15 \Rightarrow c = -3$$

$$(x^2 + 4x + 5)(x - 3) = 0$$

The last root is  $x = 3$ .

Hence the other roots are  $x = -2 - i$  and  $x = 3$

9

Let  $P(z) = z^4 - 4z^3 + az^2 + bz + 78$

Since  $3 + 2i$  is a root of  $P(z) = 0$ ,  $P(3 + 2i) = 0$ .

$$\begin{aligned}(3 + 2i)^4 - 4(3 + 2i)^3 + a(3 + 2i)^2 + b(3 + 2i) + 78 &= 0 \\ (-119 + 120i) - 4(-9 + 46i) + a(5 + 12i) + b(3 + 2i) + 78 &= 0\end{aligned}$$

Equating real parts:  $5a + 3b = 5$  ----- (1)

Equating imaginary parts:  $12a + 2b = 64$  ----- (2)

Solving (1) and (2), we get  $a = 7$  and  $b = -10$

Now  $P(z) = z^4 - 4z^3 + 7z^2 - 10z + 78$

Since coefficients of  $P(z)$  are all real,  $3 - 2i$  is also a root of  $P(z) = 0$ .

A quadratic factor of  $P(z) = [z - (3 - 2i)][z - (3 + 2i)]$

$$\begin{aligned}&= [(z - 3) + 2i][(z - 3) - 2i] \\ &= (z - 3)^2 - (2i)^2 \\ &= z^2 - 6z + 13\end{aligned}$$

Consider  $P(z) = z^4 - 4z^3 + 7z^2 - 10z + 78 = (z^2 - 6z + 13)(z^2 + cz + d)$

Comparing constants:  $78 = 13d \Rightarrow d = 6$

Comparing coefficient of  $z$ :  $-10 = -6d + 13c \Rightarrow c = 2$

Equating  $z^2 + 2z + 6 = 0$ , we have  $z = \frac{-2 \pm \sqrt{2^2 - 4(1)(6)}}{2(1)} = \frac{-2 \pm \sqrt{4(-5)}}{2}$

$$= -1 \pm \sqrt{5}i$$

Therefore, the other roots are  $3 - 2i$ ,  $-1 + \sqrt{5}i$  and  $-1 - \sqrt{5}i$ .

10

Since the coefficients of the polynomial are real,  $\sqrt{3}+i$  is a root implies that  $\sqrt{3}-i$  is also a root.

$$\begin{aligned} \text{A quadratic factor is: } & [z - (\sqrt{3} + i)][z - (\sqrt{3} - i)] \\ & = [(z - \sqrt{3}) - i][(z - \sqrt{3}) + i] \\ & = (z - \sqrt{3})^2 - (i)^2 \\ & = z^2 - 2\sqrt{3}z + 3 + 1 \\ & = z^2 - 2\sqrt{3}z + 4 \end{aligned}$$

Let  $z = k$  be the third root.

$$z^3 - 8z + a = (z^2 - 2\sqrt{3}z + 4)(z - k)$$

Comparing coefficient of  $z^2$ :

$$0 = -2\sqrt{3} - k$$

$$k = -2\sqrt{3}$$

Therefore  $z = \sqrt{3} + i$  or  $\sqrt{3} - i$  or  $-2\sqrt{3}$ .



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(a) Since the coefficients of the polynomial are real, another root is  $z = 1 - ai$ .

(b) Let  $z^3 - 2z + k = [z - (1 + ai)][z - (1 - ai)](z - c)$ , where  $c$  is a real constant.

$$\begin{aligned} z^3 - 2z + k &= [(z - 1) - ai][(z - 1) + ai](z - c) \\ &= [(z - 1)^2 - (ai)^2](z - c) \\ &= (z^2 - 2z + 1 + a^2)(z - c) \end{aligned}$$

Comparing the coefficients of  $z^2$ :

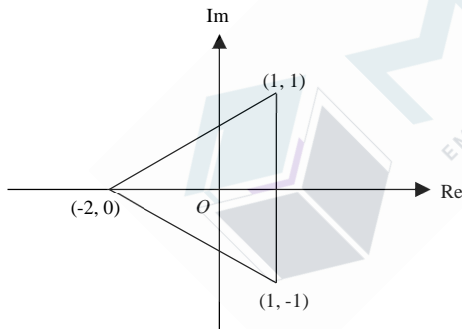
$$-c - 2 = 0 \Rightarrow c = -2$$

Comparing the coefficients of  $z$ :

$$a^2 + 1 + 2c = -2 \Rightarrow a = 1 \text{ since } a > 0$$

$$\text{So, } k = -c(1 + a^2) = 4$$

(c) Area =  $\frac{1}{2}(2)(3) = 3$  square units



12

Substitute  $z = k + ki$  into the equation

$$3(k + ki)^2 - (5 + i)(k + ki) + \alpha = 0$$

$$3k^2(1 + i)^2 - k(5 + i)(1 + i) + \alpha = 0$$

$$3k^2(1 + 2i + i^2) - k(5 + 5i + 5i + i^2) + \alpha = 0$$

$$3k^2(2i) - k(4 + 6i) + \alpha = 0 \quad \text{-----(*)}$$

$$(\alpha - 4k) + (6k^2 - 6k)i = 0$$

Comparing real and imaginary parts:

$$\alpha - 4k = 0 \quad \text{-----(1)}$$

$$6k^2 - 6k = 0 \quad \text{-----(2)}$$

$$6k(k - 1) = 0$$

$k = 1$  or  $k = 0$  (rejected since  $k$  is non-zero)

Substitute  $k = 1$  into (1)

$$\alpha = 4$$

Let the other root be  $w$

$$3z^2 - (5 + i)z + 4 = 3(z - w)(z - 1 - i)$$

Comparing constants:

$$4 = 3(-w)(-1 - i)$$

$$w = \frac{4}{3(1 + i)}$$

$$= \frac{4}{3(1 + i)} \times \frac{(1 - i)}{(1 - i)}$$

$$= \frac{2}{3} - \frac{2}{3}i$$



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$$2z + 1 = |w| \quad (1)$$

$$2w - z = 4 + 24i \quad (2)$$

From (2):  $z = 2w - 4 - 24i$

Substitute into (1):  $2(2w - 4 - 24i) + 1 = |w|$   
 $4w - 7 - 48i = |w|$

Let  $w = a + bi$

$$4(a + bi) - 7 - 48i = \sqrt{a^2 + b^2}$$

$$(4a - 7) + (4b - 48)i = \sqrt{a^2 + b^2}$$

Comparing Imaginary parts,

$$4b - 48 = 0$$

$$b = 12$$

Comparing Real parts,

$$4a - 7 = \sqrt{a^2 + b^2}$$

$$4a - 7 = \sqrt{a^2 + 12^2} \quad **$$

$$(4a - 7)^2 = a^2 + 144$$

$$15a^2 - 56a - 95 = 0$$

$$\Rightarrow a = -\frac{19}{15} \text{ or } a = 5$$

From \*\*, since  $4a - 7 =$  a positive real number,

$$\text{when } a = -\frac{19}{15}, 4a - 7 = 4\left(-\frac{19}{15}\right) - 7 < 0$$

$$\Rightarrow \text{reject } a = -\frac{19}{15}$$

$$\therefore a = 5, b = 12, z = 2(5 + 12i) - 4 - 24i = 6$$

$$\Rightarrow w = 5 + 12i, z = 6$$

14

- (a) Since  $2 + 3i$  is a root and all the coefficients are real,  $2 - 3i$  is also a root.

A quadratic factor is:

$$\begin{aligned} & [z - (2 + 3i)][z - (2 - 3i)] \\ &= [(z - 2) - 3i][(z - 2) + 3i] \\ &= (z - 2)^2 - (3i)^2 \\ &= (z^2 - 4z + 4) + 9 \\ &= z^2 - 4z + 13 \end{aligned}$$

$$z^3 - 3z^2 + kz + 13 = (z^2 - 4z + 13)(z + 1)$$

Comparing coefficient of  $z$ :  $k = 13 - 4 = 9$

The other roots are  $z = 2 - 3i$  and  $z = -1$ .

- (b) Let  $z = iw$ , then we get  $(iw)^3 - 3(iw)^2 + k(iw) + 13 = 0$   
 $\Rightarrow -iw^3 + 3w^2 + kiw + 13 = 0$

Replace  $z$  with  $iw$ ,

$$\begin{aligned} iw = 2 + 3i, & \quad iw = 2 - 3i & \text{and} & \quad iw = -1 \\ w = \frac{2 + 3i}{i} = 3 - 2i, & \quad w = \frac{2 - 3i}{i} = -3 - 2i & \text{and} & \quad w = -\frac{1}{i} = i \end{aligned}$$

15

- (a) There is only one(positive) real root in the equation  $f(x)=0$ .

Since the equation has all real coefficients, then the two other roots must be a pair of complex conjugates.

- (b) Since  $x=1-2i$  is a root of  $2x^3-7x^2+16x+c=0$ ,

$$2(-11+2i)-7(-3-4i)+16(1-2i)+c=0$$

$$15+c=0$$

$$\therefore c=-15 \text{ (shown)}$$

- (c) Since all the coefficients are real,  $x=1+2i$  is another root of  $2x^3-7x^2+16x+c=0$ .

$$2x^3-7x^2+16x-15=0$$

$$[x-(1+2i)][x-(1-2i)](2x-k)=0$$

$$[(x-1)+2i][(x-1)-2i](2x-k)=0$$

$$[x^2-2x+5](2x-k)=0$$

Comparing the coefficient of constant term (or by long division),

$$-5k=-15 \Rightarrow k=3$$

Therefore, the last root is  $x=\frac{3}{2}$

The roots are  $x=1+2i$ ,  $x=1-2i$ ,  $x=\frac{3}{2}$

- (d)  $2x^3-7x^2+16x-15=0$

Replace  $x$  with  $\frac{1}{w}$

$$2\left(\frac{1}{w}\right)^3-7\left(\frac{1}{w}\right)^2+16\left(\frac{1}{w}\right)-15=0$$

$$2-7w+16w^2-15w^3=0$$

Hence, the roots are  $\frac{1}{w}=1+2i$ ;  $\frac{1}{w}=1-2i$ ;  $\frac{1}{w}=\frac{3}{2}$

$$w=\frac{1}{1+2i}=\frac{1}{5}-\frac{2}{5}i$$

$$w=\frac{1}{1-2i}=\frac{1}{5}+\frac{2}{5}i$$

$$w=\frac{2}{3}$$