1

(i) $y = vx^2$

$$\frac{dy}{dx} = v(2x) + \frac{dv}{dx}(x^2)$$

$$x\frac{dy}{dx} = x^3 + xy + 2y$$

$$x\left(2vx + x^2\frac{dv}{dx}\right) = x^3 + x(vx^2) + 2(vx^2)$$

$$2vx^2 + x^3\frac{dv}{dx} = x^3 + vx^3 + 2vx^2$$

$$x^3\frac{dv}{dx} = x^3 + vx^3$$

$$\frac{dv}{dx} = 1 + v \qquad \text{(shown)}$$

(ii) $\frac{dv}{dx} = 1 + v$

$$\frac{1}{1+v}\frac{dv}{dx} = 1$$

$$\int \frac{1}{1+v}\,dv = \int 1\,dx$$

$$\ln|1+v| = x + c$$

$$|1+v| = e^{x+c}$$

$$1 + v = \pm e^{x+c}$$

$$1 + v = Ae^x \quad \text{where } A = \pm e^c$$

$$1 + \frac{y}{x^2} = Ae^x$$

When $x = 1,\ y = 1$

$$1 + \frac{1}{1^2} = Ae$$

$$A = \frac{2}{e}$$

$$1 + \frac{y}{x^2} = \left(\frac{2}{e}\right)e^x$$

$$1 + \frac{y}{x^2} = 2e^{x-1}$$

$$\frac{y}{x^2} = 2e^{x-1} - 1$$

$$y = x^2\left(2e^{x-1} - 1\right)$$

2

(i) $v = 4x - y$

$$\frac{dv}{dx} = 4 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = (4x - y + 2)^2$$

$$\frac{dv}{dx} = 4 - (4x - y + 2)^2$$

$$\frac{dv}{dx} = 4 - (v + 2)^2$$

(ii) $$\frac{dv}{dx} = 4 - (v + 2)^2$$

$$\frac{1}{4 - (v+2)^2} \frac{dv}{dx} = 1$$

$$\int \frac{1}{4 - (v+2)^2} dv = \int 1 dx$$

$$\frac{1}{2(2)} \ln\left|\frac{2 + (v+2)}{2 - (v+2)}\right| = x + C$$

$$\ln\left|\frac{v+4}{-v}\right| = 4x + 4C$$

$$1 + \frac{4}{v} = \pm e^{4x+4C}$$

$$1 + \frac{4}{v} = Ae^{4x}, \text{ where } A = \pm e^{4C}$$

When $x = 0,\ y = -2,\ \therefore v = 0 - (-2) = 2$.

Hence $1 + \frac{4}{2} = Ae^0 \Rightarrow A = 3$

$$1 + \frac{4}{v} = 3e^{4x}$$

$$v = \frac{4}{3e^{4x} - 1}$$

$$4x - y = \frac{4}{3e^{4x} - 1}$$

$$y = 4x - \frac{4}{3e^{4x} - 1}$$

3

(a) $y = z\sec x$

$$\frac{dy}{dx} = z\sec x\tan x + \frac{dz}{dx}\sec x$$

$$\pi\frac{dy}{dx} + y(3 - \pi\tan x) = 0$$

$$\pi\left(z\sec x\tan x + \frac{dz}{dx}\sec x\right) + z\sec x(3 - \pi\tan x) = 0$$

$$\pi z\sec x\tan x + \pi\frac{dz}{dx}\sec x + 3z\sec x - \pi z\sec x\tan x = 0$$

$$\pi\frac{dz}{dx}\sec x + 3z\sec x = 0$$

$$\pi\frac{dz}{dx}\sec x = -3z\sec x$$

$$\frac{dz}{dx} = \frac{-3z}{\pi} \quad \text{(shown)}$$

$$\frac{dz}{dx} = \frac{-3z}{\pi}$$

$$\frac{1}{z}\frac{dz}{dx} = \frac{-3}{\pi}$$

$$\int \frac{1}{z}\,dz = \int \frac{-3}{\pi}\,dx$$

$$\ln|z| = \frac{-3}{\pi}x + C$$

$$z = \pm e^{\frac{-3}{\pi}x + C}$$

$$z = Ae^{\frac{-3}{\pi}x} \quad \text{where } A = \pm e^{C}$$

$$\frac{y}{\sec x} = Ae^{\frac{-3}{\pi}x}$$

$$y\cos x = Ae^{\frac{-3}{\pi}x}$$

When $x = \frac{\pi}{3}$ and $y = 2$,

$$2\cos\frac{\pi}{3} = Ae^{\frac{-3}{\pi}\left(\frac{\pi}{3}\right)}$$

$$A = e$$

$$\therefore y\cos x = e\left(e^{\frac{-3}{\pi}x}\right)$$

$$y\cos x = e^{1 - \frac{3}{\pi}x} \quad \text{where } a = 1 \text{ and } b = -\frac{3}{\pi}$$

(b) $y = \dfrac{e^{1-\frac{3}{\pi}x}}{\cos x}$

For vertical asymptotes, consider $\cos x = 0$

The asymptotes closest to *y*-axis are $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$.

4

(a) $\dfrac{dP}{dt}=\dfrac{1}{26}P(13-2P)$

$$\frac{1}{P(13-2P)}\frac{dP}{dt}=\frac{1}{26}$$

$$\int \frac{1}{P(13-2P)}\ dP=\int \frac{1}{26}\ dt$$

<u>Method ①: Partial Fractions</u>

$$\frac{1}{P(13-2P)}=\frac{M}{P}+\frac{N}{13-2P}$$

$$1=M(13-2P)+NP$$

When $P=0$, $M=\dfrac{1}{13}$

When $P=\dfrac{13}{2}$, $N=\dfrac{2}{13}$

$$\frac{1}{P(13-2P)}=\frac{1}{13P}+\frac{2}{13(13-2P)}$$

$$=\frac{1}{13}\left(\frac{1}{P}+\frac{2}{13-2P}\right)$$

$$\frac{1}{13}\int \frac{1}{P}+\frac{2}{(13-2P)}\, dP=\frac{1}{26}\int 1\, dt$$

$$\int \frac{1}{P}\, dP-\int \frac{-2}{(13-2P)}\, dP=\frac{1}{2}\int 1\, dt$$

$$\ln|P|-\ln|13-2P|=\frac{1}{2}t+C$$

$$\ln\left|\frac{P}{13-2P}\right|=\frac{1}{2}t+C$$

$$\left|\frac{P}{13-2P}\right|=e^{\frac{1}{2}t+C}$$

$$\frac{P}{13-2P}=\pm e^{\frac{1}{2}t+C}$$

$$\frac{P}{13-2P}=Ae^{\frac{1}{2}t},\ \text{where } A=\pm e^{C}$$

When $t=0$, $P=2$,

$$\frac{2}{13-4}=Ae^{0}$$

$$A=\frac{2}{9}$$

<u>Method ②: Complete the square</u>

$$P(13-2P)=13P-2P^2$$

$$=-2\left[P^2-\frac{13}{2}P\right]$$

$$=-2\left[\left(P-\frac{13}{4}\right)^2-\left(\frac{13}{4}\right)^2\right]$$

$$-\frac{1}{2}\int \frac{1}{\left(P-\frac{13}{4}\right)^2-\left(\frac{13}{4}\right)^2}\, dP=\frac{1}{26}\int 1\, dt$$

$$-\frac{1}{2}\frac{1}{2\left(\frac{13}{4}\right)}\ln\left|\frac{P-\frac{13}{4}-\frac{13}{4}}{P-\frac{13}{4}+\frac{13}{4}}\right|=\frac{1}{26}\int 1\, dt$$

$$-\frac{1}{13}\ln\left|\frac{P-\frac{13}{2}}{P}\right|=\frac{1}{26}\int 1\, dt$$

$$\ln\left|\frac{2P-13}{2P}\right|=-\frac{1}{2}t+C$$

$$\left|\frac{2P-13}{2P}\right|=e^{-\frac{1}{2}t+C}$$

$$\frac{2P-13}{2P}=\pm e^{-\frac{1}{2}t+C}$$

$$\frac{2P-13}{2P}=Ae^{-\frac{1}{2}t},\ \text{where } A=\pm e^{C}$$

When $t=0$, $P=2$,

$$\frac{2(2)-13}{2(2)}=Ae^{0}$$

$$A=-\frac{9}{4}$$

$$\frac{P}{13-2P}=\frac{2}{9}e^{\frac{1}{2}t}$$
$$9P=2e^{\frac{1}{2}t}(13-2P)$$
$$9P=26e^{\frac{1}{2}t}-4Pe^{\frac{1}{2}t}$$
$$9P+4Pe^{\frac{1}{2}t}=26e^{\frac{1}{2}t}$$
$$P\left(9+4e^{\frac{1}{2}t}\right)=26e^{\frac{1}{2}t}$$
$$P=\frac{26e^{\frac{1}{2}t}}{9+4e^{\frac{1}{2}t}}\cdot\frac{e^{-\frac{1}{2}t}}{e^{-\frac{1}{2}t}}$$
$$P=\frac{26}{9e^{-\frac{1}{2}t}+4} \quad \text{(shown)}$$

$$\frac{2P-13}{2P}=-\frac{9}{4}e^{-\frac{1}{2}t}$$
$$8P-52=-18Pe^{-\frac{1}{2}t}$$
$$8P+18Pe^{-\frac{1}{2}t}=52$$
$$P\left(8+18Pe^{-\frac{1}{2}t}\right)=52$$
$$P=\frac{52}{8+18Pe^{-\frac{1}{2}t}}$$
$$P=\frac{26}{9e^{-\frac{1}{2}t}+4} \quad \text{(shown)}$$

(b) When $P=4$,

$$4=\frac{26}{9e^{-\frac{1}{2}t}+4}$$
$$4\left(9e^{-\frac{1}{2}t}+4\right)=26$$
$$9e^{-\frac{1}{2}t}=\frac{5}{2}$$
$$e^{-\frac{1}{2}t}=\frac{5}{18}$$
$$-\frac{1}{2}t=\ln\left(\frac{5}{18}\right)$$
$$t=-2\ln\left(\frac{5}{18}\right)$$
$$t=2.56$$

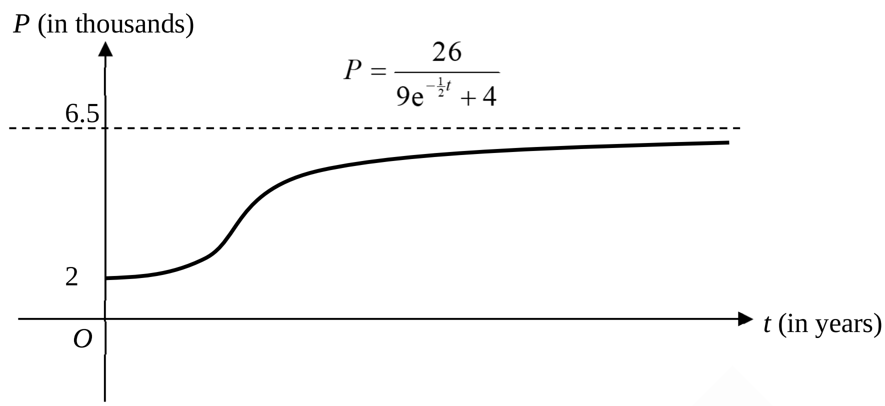
It takes 2.56 months for the number of people who downloaded Ginseng Impact to double since the launch.

(c) As $t\to\infty$, $e^{-\frac{1}{2}t}\to 0$,

$$P\to\frac{26}{4}$$

Number of people that downloaded Ginseng Impactful in the long run is $\frac{26}{4}(1000)=6500$.

(d)



5

(a) $y = ux^2$

$$\frac{dy}{dx} = u(2x) + x^2\frac{du}{dx}$$

$$\frac{dy}{dx} - \frac{2y}{x} = x^3$$

$$2ux + x^2\frac{du}{dx} - \frac{2(ux^2)}{x} = x^3$$

$$2ux + x^2\frac{du}{dx} - 2ux = x^3$$

$$x^2\frac{du}{dx} = x^3$$

Since $x \neq 0$, $\frac{du}{dx} = x$ (shown)

$$u = \frac{1}{2}x^2 + C$$

$$\frac{y}{x^2} = \frac{1}{2}x^2 + C$$

$$y = \frac{1}{2}x^4 + Cx^2$$

(b) $\dfrac{dN}{dt} \propto N$

$\dfrac{dN}{dt} = kN, \quad k > 0$

When $t = 0$, $N = 5000$, $\dfrac{dN}{dt} = 200$,

$\dfrac{dN}{dt} = kN$

$200 = 5000k$

$k = \dfrac{1}{25}$

$\dfrac{dN}{dt} = \dfrac{1}{25}N$

$\dfrac{1}{N}\dfrac{dN}{dt} = \dfrac{1}{25}$

$\displaystyle\int \frac{1}{N}\,dN = \int \frac{1}{25}\,dt$

$\ln|N| = \dfrac{1}{25}t + C$

$|N| = e^{\frac{t}{25}+C}$

$N = \pm e^{C}e^{\frac{t}{25}}$

$N = Ae^{\frac{t}{25}} \qquad$ where $A = \pm e^{C}$

When $t = 0, N = 5000$,

$5000 = Ae^{\frac{0}{25}}$

$A = 5000$

$\therefore N = 5000e^{\frac{t}{25}}$

When $t = 50$,

$N = 5000e^{\frac{50}{25}}$

$= 36900 \quad (3\text{ s.f.})$

(c) $\dfrac{\mathrm{d}N}{\mathrm{d}t} = kN(\ln M - \ln N)$

$$\int \frac{1}{N(\ln M - \ln N)}\,\mathrm{d}N = \int k\,\mathrm{d}t$$

$$-\int \frac{-\frac{1}{N}}{(\ln M - \ln N)}\,\mathrm{d}N = \int k\,\mathrm{d}t$$

$$-\ln\left|\ln M - \ln N\right| = kt + D$$

$$\ln\left|\ln M - \ln N\right| = -kt - D$$

$$\left|\ln M - \ln N\right| = \mathrm{e}^{-kt-D}$$

$$\ln M - \ln N = \pm\mathrm{e}^{-kt-D}$$

$$\ln\left(\frac{M}{N}\right) = B\mathrm{e}^{-kt} \qquad \text{where } B = \pm\mathrm{e}^{-D}$$

$$\frac{M}{N} = \mathrm{e}^{B\mathrm{e}^{-kt}}$$

$$N = M\mathrm{e}^{-B\mathrm{e}^{-kt}}$$

As $t \to \infty$,

$\mathrm{e}^{-kt} \to 0, \quad \mathrm{e}^{-B\mathrm{e}^{-kt}} \to \mathrm{e}^{-B(0)} \to 1, \quad N \to M$.

Regardless of the initial population of the bacteria, the number of bacteria always tends towards M eventually.

6

(a) $\dfrac{d\theta}{dt} \propto -(\theta - 32)$

$\dfrac{d\theta}{dt} = -k(\theta - 32)$ for some positive constant

$\displaystyle\int \frac{1}{\theta - 32}\, d\theta = -k\int 1\, dt$

Method ①:

Since pie is cooling down, $\theta \geq 32$

$\theta - 32 > 0$

$\ln(\theta - 32) = -kt + C$

$\theta - 32 = e^{-kt+C}$

$\theta - 32 = Ae^{-kt}$, where $A = e^{C}$

$\theta = 32 + Ae^{-kt}$

Method ②:

$\ln|\theta - 32| = -kt + C$

$|\theta - 32| = e^{-kt+C}$

$\theta - 32 = \pm e^{-kt+C}$

$\theta - 32 = Ae^{-kt}$, where $A = \pm e^{C}$

$\theta = 32 + Ae^{-kt}$

When $t = 0$, $\theta = 200$

$200 = 32 + A$

$A = 168$

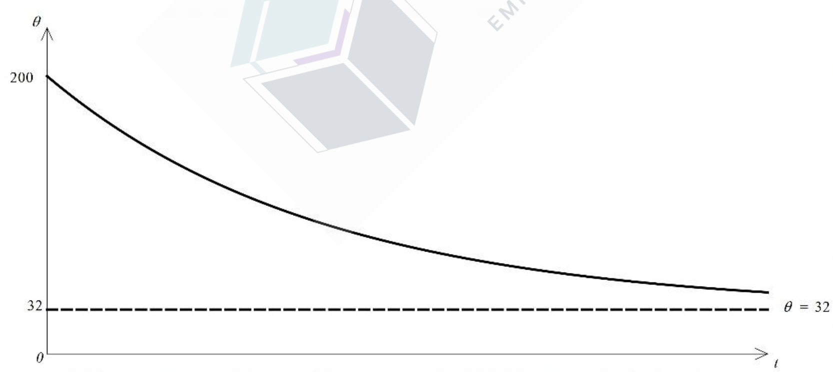
$\therefore \theta = 32 + 168e^{-kt}$

When $t = 15$, $\theta = 180 = 32 + 168e^{-15k}$

$e^{-15k} = \dfrac{148}{168} = \dfrac{37}{42}$

$e^{-k} = \left(\dfrac{37}{42}\right)^{1/15}$ $\qquad \therefore \theta = 32 + 168\left(\dfrac{37}{42}\right)^{t/15}$

(b)



(c) When $\theta = 60$, $60 = 32 + 168\left(\dfrac{37}{42}\right)^{t/15}$

Using GC, t = 212.04 (5 s.f.)

Alternatively, $60 = 32 + 168\left(\dfrac{37}{42}\right)^{t/15} \Rightarrow t = \dfrac{15\ln\left(1/6\right)}{\ln\left(37/42\right)} \approx 212.04$

= 212 mins (nearest min), equivalent to 3h 32 mins.

To safely store the pie, t < 212 mins. Latest time to keep the pie is 4.32 pm.

7

$$\frac{dv}{dt} = 9.8 - R$$

$$\frac{dv}{dt} = 9.8 - kv \qquad (\text{Since } R \propto v \Rightarrow R = kv, \text{ where } k > 0)$$

$$\int \frac{1}{9.8 - kv}\, dv = \int 1\, dt$$

$$-\frac{1}{k}\int \frac{-k}{9.8 - kv}\, dv = \int 1\, dt$$

Method ①:

Since $9.8 - kv > 0$

$$-\frac{1}{k}\ln(9.8 - kv) = t + C$$

$$\ln(9.8 - kv) = -kt - kC$$

$$9.8 - kv = e^{-kt-kC}$$

$$9.8 - kv = Ae^{-kt}, \text{ where } A = e^{-kC}$$

Method ②:

$$-\frac{1}{k}\ln|9.8 - kv| = t + C$$

$$\ln|9.8 - kv| = -kt - kC$$

$$|9.8 - kv| = e^{-kt-kC}$$

$$9.8 - kv = \pm e^{-kt-kC}$$

$$9.8 - kv = Ae^{-kt}, \text{ where } A = \pm e^{-kC}$$

When $t = 0,\ v = 0$ (helicopter is stationary)

$$9.8 - k(0) = Ae^{0}$$

$$A = 9.8$$

$$9.8 - kv = 9.8e^{-kt}$$

$$kv = 9.8 - 9.8e^{-kt}$$

$$v = \frac{9.8\left(1 - e^{-kt}\right)}{k}$$

As $t \to \infty,\ e^{-kt} \to 0,\ v \to \frac{9.8}{k}$

The terminal velocity of the parachutist is $\frac{9.8}{k}$ ms^{-1}.

8

(a) $\dfrac{dv}{dt} = 8 - kv\,,\, k > 0$

$$\int \frac{1}{8-kv}\,dv = \int 1\,dt$$

$$-\frac{1}{k}\int \frac{-k}{8-kv}\,dv = \int 1\,dt$$

$$-\frac{1}{k}\ln|8-kv| = t + c$$

$$\ln|8-kv| = -kt - kc$$

$$|8-kv| = e^{-kt-kc}$$

$$8-kv = \pm e^{-kt-kc}$$

$$8-kv = Ae^{-kt} \quad \text{where } A = \pm e^{-kc}$$

$$kv = 8 - Ae^{-kt}$$

$$v = \frac{1}{k}\left(8 - Ae^{-kt}\right)$$

When $t = 0,\ v = 0$

$$\frac{1}{k}\left(8 - Ae^{0}\right) = 0$$

$$A = 8$$

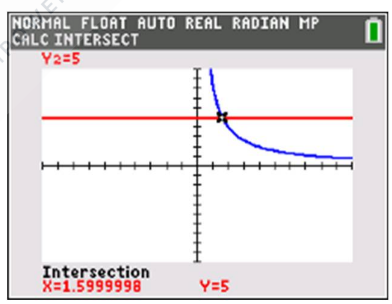
$$\therefore v = \frac{1}{k}\left(8 - 8e^{-kt}\right)$$

When $t = 10,\ v = 5$

$$5 = \frac{1}{k}\left(8 - 8e^{-10k}\right)$$

Using GC, $k = 1.6$

$$\therefore v = \frac{1}{1.6}\left(8 - 8e^{-1.6t}\right) = 5\left(1 - e^{-1.6t}\right)$$



(b) In the long run, as $t \to \infty,\ e^{-1.6t} \to 0,\ v \to 5$ gallons

(c) $\dfrac{dv}{dt} = 8 - v^2$

(d) Let T hours be the time that elapsed.

$$\frac{dv}{dt} = 8 - v^2$$

$$\int_{4.5}^{3} \frac{1}{8-v^2}\,dv = \int_0^T 1\,dt$$

$$0.36195 = T$$

Time that elapsed $= 0.36195 \times 60 = 22$ min (correct to nearest minutes)

9

(a) $\dfrac{dN}{dt} = \text{rate of increase} - \text{death rate}$

rate of increase $\propto \dfrac{1}{N^2}$

rate of increase $= \dfrac{A}{N^2}$, where $A > 0$

death rate $\propto N$

death rate $= BN$, where $B > 0$

$$\therefore \frac{dN}{dt} = \frac{A}{N^2} - BN$$

When $N = 3$, $\dfrac{dN}{dt} = 0$ (number of rodents remain constant)

$$\frac{A}{3^2} - 3B = 0$$

$$A - 27B = 0$$

$A = 27B$ or $B = \dfrac{1}{27}A$

$$\frac{dN}{dt} = \frac{27B}{N^2} - BN = \frac{B\left(27 - N^3\right)}{N^2} = \frac{k\left(27 - N^3\right)}{N^2}\text{, where } k = B$$

or

$$\frac{dN}{dt} = \frac{A}{N^2} - \frac{A}{27}N = \frac{27A - AN^3}{27N^2} = \frac{A}{27}\frac{\left(27 - N^3\right)}{N^2} = \frac{k\left(27 - N^3\right)}{N^2}\text{, where } k = \frac{A}{27}$$

(b) $\dfrac{dN}{dt} = \dfrac{k(27-N^3)}{N^2}$

$$\frac{N^2}{27-N^3}\frac{dN}{dt} = k$$

$$\int \frac{N^2}{27-N^3}\, dN = \int k\ dt$$

$$-\frac{1}{3}\int \frac{-3N^2}{27-N^3}\, dN = \int k\ dt$$

$$-\frac{1}{3}\ln\left|27-N^3\right| = kt + D$$

$$\left|27-N^3\right| = -3kt - 3D$$

$$27-N^3 = \pm e^{-3kt-3D}$$

$$27-N^3 = Me^{-3kt}, \text{ where } M = \pm e^{-3D}$$

When $t = 0$, $N = 5$,

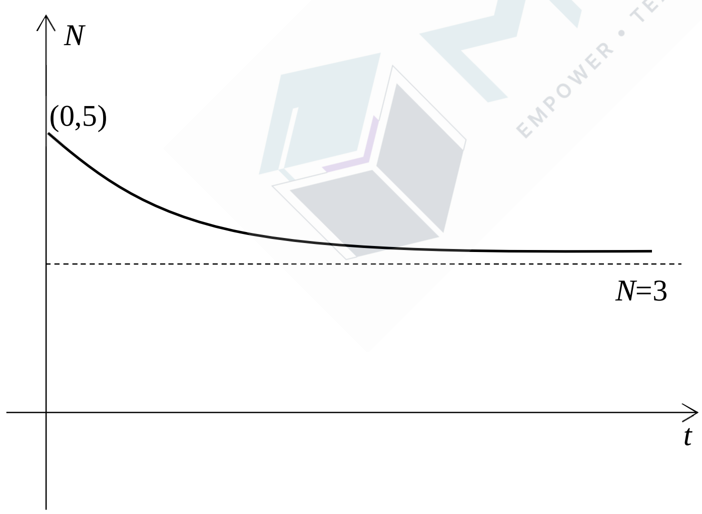
$$27-5^3 = M$$

$$M = -98$$

$$27-N^3 = -98e^{-3kt}$$

$$N^3 = 27+98e^{-3kt}$$

(c) The population of rodents will decrease and eventually approach 300 in the long run.



10

(i) $\dfrac{dQ}{dt} = \dfrac{dQ_{in}}{dt} - \dfrac{dQ_{out}}{dt}$

$\dfrac{dQ_{in}}{dt} = k$

$\dfrac{dQ_{out}}{dt} \propto \sqrt{Q}$

$\dfrac{dQ_{out}}{dt} = c\sqrt{Q}$, where $c > 0$

$\dfrac{dQ}{dt} = k - c\sqrt{Q}$

Starting with a new clean tank:

When $t = 0$, $Q = 0$, $\dfrac{dQ}{dt} = 5$

$\dfrac{dQ}{dt} = k - c\sqrt{Q}$

$5 = k - c\sqrt{0}$

$k = 5$

$\dfrac{dQ}{dt} = 5 - c\sqrt{Q}$

With filter in a new clean tank, level of pollution stabilizes at 75 units:

As $t \to \infty$, $Q \to 75$, $\dfrac{dQ}{dt} \to 0$,

$\dfrac{dQ}{dt} = 5 - c\sqrt{Q}$

$0 = 5 - c\sqrt{75}$

$5\sqrt{3}c = 5$

$c = \dfrac{1}{\sqrt{3}}$

$\therefore \dfrac{dQ}{dt} = 5 - \sqrt{\dfrac{Q}{3}}$ (shown)

(ii) $x=\sqrt{\frac{Q}{3}}$

$x^2=\frac{Q}{3}$

$Q=3x^2$

$\frac{dQ}{dt}=6x\frac{dx}{dt}$

$\frac{dQ}{dt}=5-\sqrt{\frac{Q}{3}}$

$6x\frac{dx}{dt}=5-x$ (shown)

(iii) $6x\frac{dx}{dt}=5-x$

$\frac{6x}{5-x}\frac{dx}{dt}=1$

$\int\frac{6x}{5-x}dx=\int dt$

$6\int -1+\frac{5}{5-x}dx=\int dt$

$-x-5\ln|5-x|=\frac{1}{6}t+c$

$$x+\ln|5-x|^5=\frac{-t}{6}-c$$

$$\ln|5-x|^5=-x-\frac{t}{6}-c$$

$$|5-x|^5=\mathrm{e}^{-x-\frac{t}{6}-c}$$

$$(5-x)^5=\pm\mathrm{e}^{-c}\,\mathrm{e}^{-x}\mathrm{e}^{-\frac{t}{6}}$$

$$(5-x)^5\mathrm{e}^{x}=A\mathrm{e}^{\frac{-t}{6}}\quad\text{where } A=\pm\mathrm{e}^{-c}$$

$$\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=A\mathrm{e}^{\frac{-t}{6}}$$

When $t = 0$, $Q = 0$:

$$\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=A\mathrm{e}^{\frac{-t}{6}}$$

$$\left(5-\sqrt{\frac{0}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{0}{3}}}=A\mathrm{e}^{0}$$

$$A=5^5=3125$$

$$\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=3125\mathrm{e}^{\frac{-t}{6}}$$

Therefore, $a = 5$, $b = 5$, $m = 3125$, and $p=\frac{-1}{6}$.

(iv) When $Q = 48$,

$$\left(5-\sqrt{\frac{48}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{48}{3}}}=3125\mathrm{e}^{\frac{-t}{6}}$$

$$t=-6\ln\left(\frac{\mathrm{e}^4}{3125}\right)=24.3\text{ days (3 s.f.)}$$

Without a filter, the pollutant level would reach 48 units in $t = 48 / 5 = 9.6$ days.
Therefore the filter is effective.

11

(i) $\frac{dv}{dt}+\frac{6\pi\eta R}{m}v=g$

Substituting all the values, $\eta=1.3806, R=0.05, m=0.2, g=9.780$ gives

$$\frac{dv}{dt}+6.5059v=9.780$$

$$\frac{dv}{dt}=9.780-6.5059v$$

$$\frac{1}{9.780-6.5059v}\frac{dv}{dt}=1$$

$$\int\frac{1}{9.780-6.5059v}\,dv=\int 1\,dt$$

$$\int\frac{1}{9.780-6.5059v}\,dv=t+c$$

$$-\frac{1}{6.5059}\int\frac{-6.5059}{9.780-6.5059v}\,dv=t+c$$

$$\frac{\ln|9.780-6.5059v|}{-6.5059}=t+c$$

$$\ln|9.780-6.5059v|=-6.5059t-6.5059c$$

$$|9.780-6.5059v|=e^{-6.5059t-6.5059c}$$

$$9.780-6.5059v=\pm e^{-6.5059t-6.5059c}$$

$$9.780-6.5059v=Ae^{-6.5059t},\quad A=\pm e^{-6.5059c}$$

At $t=0, v=0$, (ball is released from rest)

$$9.780-6.5059(0)=Ae^{-6.5059(0)}$$

$$A=9.780$$

$$\therefore 9.780-6.5059v=9.780e^{-6.5059t}$$

$$6.5059v=9.780-9.780e^{-6.5059t}$$

$$=9.780\left(1-e^{-6.5059t}\right)$$

$$v=1.5032\left(1-e^{-6.5059t}\right)$$

$$v=1.50\left(1-e^{-6.51t}\right)\qquad \text{(3 s.f.)}$$

(ii) As $t\to\infty, e^{-6.5059t}\to 0$, $v\to 1.5032$

The ball approaches a velocity of 1.50 ms^{-1} after a long time.

(iii) **<u>Method ①:</u>**

$$v = \frac{dx}{dt}$$

$$\int v \, dt = \int \frac{dx}{dt} \, dt$$

$$x = \int v \, dt$$

$$x = \int_0^{30} 1.5032\left(1 - e^{-6.5059t}\right) dt \quad \text{(using GC)}$$

$$= 44.865 \approx 44.9$$

The distance covered by the ball after 30 seconds is 44.9 m.

Method ②:

$$v = \frac{dx}{dt} = 1.5032\left(1 - e^{-6.5059t}\right)$$

$$x = \int 1.5032\left(1 - e^{-6.5059t}\right) dt$$

$$= 1.5032\int\left(1 - e^{-6.5059t}\right) dt$$

$$= 1.5032\left[\int 1dt \, dt - \int e^{-6.5059t} \, dt\right]$$

$$= 1.5032\left[\int 1dt \, dt + \frac{1}{6.5059}\int -6.5059e^{-6.5059t} \, dt\right]$$

$$= 1.5032\left[t + \frac{e^{-6.5059t}}{6.5059}\right] + c$$

When $t = 0, x = 0 \Rightarrow c = -\frac{1.5039}{6.5059}$

When $t = 30$,

$$x = 1.5032\left[30 + \frac{e^{-6.5059(30)}}{6.5059}\right] - \frac{1.5032}{6.5059}$$

$$= 44.9 \text{ (3s.f.)}$$

12

(a) y decreases at a rate proportional to y implies $\frac{\mathrm{d}y}{\mathrm{d}t} \propto y$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -ay$, where $a > 0$

Method ①:

$$x^2 + y^2 = 8^2$$

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= -\frac{y}{x}(-ay)$$

$$= \frac{ay^2}{x}$$

$$= \frac{a\left(64 - x^2\right)}{x}$$

$$= \frac{k\left(64 - x^2\right)}{x}, \text{ where } k = a \text{ (shown)}$$

Method ②:

$$x^2 + y^2 = 8^2$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y(-ay) = 0$$

$$x\frac{\mathrm{d}x}{\mathrm{d}t} - ay^2 = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{ay^2}{x}$$

$$= \frac{a\left(64 - x^2\right)}{x}$$

$$= \frac{k\left(64 - x^2\right)}{x}, \text{ where } k = a \text{ (shown)}$$

(b) Given that $k = 3$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3\left(64 - x^2\right)}{x}$$

$$\frac{x}{64 - x^2}\frac{\mathrm{d}x}{\mathrm{d}t} = 3$$

$$\int \frac{x}{64 - x^2}\,\mathrm{d}x = \int 3\,\mathrm{d}t$$

$$-\frac{1}{2}\int \frac{-2x}{64 - x^2}\,\mathrm{d}x = 3t + C$$

Method ①:

Since $x < 8$, $64 - x^2 > 0$

$$-\frac{1}{2}\ln\left(64 - x^2\right) = 3t + C$$

$$\ln\left(64 - x^2\right) = -6t - 2C$$

$$64 - x^2 = \mathrm{e}^{-6t-2C}$$

$$64 - x^2 = A\mathrm{e}^{-6t}, \text{ where } A = \mathrm{e}^{-2C}$$

Method ②:

$$-\frac{1}{2}\ln\left|64 - x^2\right| = 3t + C$$

$$\ln\left|64 - x^2\right| = -6t - 2C$$

$$\left|64 - x^2\right| = \mathrm{e}^{-6t-2C}$$

$$64 - x^2 = \pm\mathrm{e}^{-6t-2C}$$

$$64 - x^2 = A\mathrm{e}^{-6t}, \text{ where } A = \pm\mathrm{e}^{-2C}$$

When $t = 0$, $x = 4$,

$$64 - 4^2 = A\mathrm{e}^{-6(0)}$$

$$A = 48$$

$$\frac{dx}{dt}=\frac{3\left(64-x^2\right)}{x}$$

$$\frac{x}{64-x^2}\frac{dx}{dt}=3$$

$$\int\frac{x}{64-x^2}\,dx=\int 3\,dt$$

$$-\tfrac{1}{2}\ln(64-x^2)=3t+C \qquad \text{(since } x\le 8)$$

$$x^2=64-Ae^{-6t} \qquad \text{where } A=e^{-2C}$$

$$x=\sqrt{64-Ae^{-6t}}$$

When $t=0,\ x=4$:

$$4=\sqrt{64-A}$$

$$A=48$$

$$\therefore x=\sqrt{64-48e^{-6t}}$$

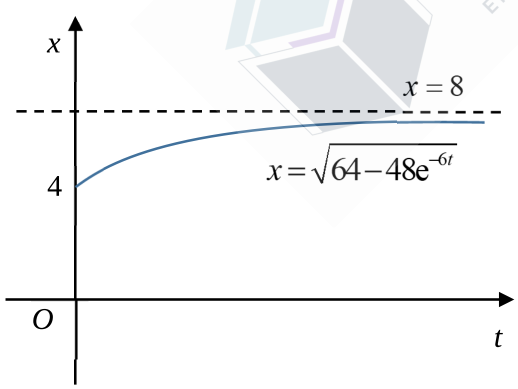
When $y=3$, $x=\sqrt{64-3^2}=\sqrt{55}$

$$\sqrt{55}=\sqrt{64-48e^{-6t}}$$

$$e^{-6t}=\frac{9}{48}$$

$t=0.279=0.3$ s (1 d.p.)

(c)



(d) Based on Jim's conjecture, the rod will never be flat on the ground; thus not appropriate.

13

(i) $\frac{dQ}{dt} = \frac{dQ_{in}}{dt} - \frac{dQ_{out}}{dt}$

$\frac{dQ_{in}}{dt} \propto \frac{1}{Q}$ $\qquad\qquad$ $\frac{dQ_{out}}{dt} \propto Q$

$\frac{dQ_{in}}{dt} = \frac{a}{Q},\ a > 0$ $\qquad\qquad$ $\frac{dQ_{out}}{dt} = bQ,\ b > 0$

Rate of change of amount of glucose, $\frac{dQ}{dt} = \frac{dQ_{in}}{dt} - \frac{dQ_{out}}{dt}$

$$\frac{dQ}{dt} = \frac{a}{Q} - bQ$$

The amount of glucose in the bloodstream remains constant when $Q = 4$ implies that

When $Q = 4, \frac{dQ}{dt} = 0$

$$\frac{a}{4} - 4b = 0$$

$$a = 16b \qquad \text{or} \qquad b = \frac{1}{16}a$$

$$\frac{dQ}{dt} = \frac{16b}{Q} - bQ \qquad\qquad \frac{dQ}{dt} = \frac{a}{Q} - \frac{aQ}{16}$$

$$= b\left(\frac{16-Q^2}{Q}\right) \qquad \text{or} \qquad = a\left(\frac{16-Q^2}{16Q}\right)$$

$$= k\left(\frac{16-Q^2}{Q}\right), \text{ where } k = b \qquad\qquad = k\left(\frac{16-Q^2}{Q}\right), \text{ where } k = \frac{a}{16}$$

(ii) $\frac{dQ}{dt} = k\left(\frac{16-Q^2}{Q}\right)$

$$\frac{Q}{16-Q^2}\frac{dQ}{dt} = k$$

$$\int \frac{Q}{16-Q^2}\, dQ = \int k\ dt$$

$$-\frac{1}{2}\int \frac{-2Q}{16-Q^2}\, dQ = \int k\ dt$$

$$-\frac{1}{2}\ln\left|16-Q^2\right| = kt + C$$

$$\ln\left|16-Q^2\right| = -2kt - 2C$$

$$\left|16-Q^2\right| = e^{-2kt-2C}$$

$$16-Q^2 = \pm e^{-2kt-2C}$$

$$16-Q^2 = De^{-2kt},\ \text{where } D = \pm e^{-2C}$$

When $t=0,\ Q=7$

$$D=7^2-16$$
$$D=-33$$
$$16-Q^2=-33e^{-2kt}$$

When $t=15, Q=6.8$

$$16-6.8^2=-33e^{-30k}$$
$$e^{-30k}=0.91636$$
$$-30k=\ln 0.91636$$
$$-30k=-0.087342$$
$$k=0.0029114$$

$$16-Q^2=-33e^{-2(0.0029114)t}$$
$$Q^2=16+33e^{-0.0058228007t}$$
$$Q^2=16+33e^{-0.00582t},\ \text{where } A=33, B=-0.00582 \quad \text{(3.s.f)}$$

(iii) When t = 60,

$$Q^2=16+33e^{-0.0058228(60)}$$
$$Q=6.267\approx 6.27 \text{ (3 s.f.)}$$

(iv) Since the glucose level after 1 hr is 6.27 mmol/L which is not within the normal range, hence the clinical trial is not as effective as it claims.

(v) When $t\to\infty$, $e^{-0.00582t}\to 0$, $Q^2\to 16$. Hence $Q\to 4$.

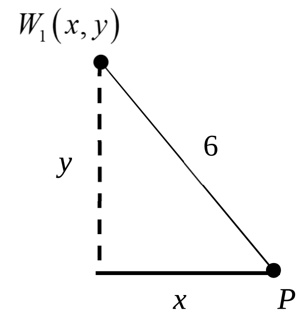
Hence the amount of glucose in the patient's bloodstream approaches 4 mmol/L in the long run.

The model might not be feasible in the long run as there may be other external factors such as consumption of food, that might affect the glucose level in the patient's bloodstream.

14

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives the gradient of the tangent of C.

Using Pythagoras Theorem, $x^2 + y^2 = 6^2 \Rightarrow x = \sqrt{36 - y^2}$ (since $x > 0$)



At W_1 with coordinates (x, y), absolute value of the gradient of the tangent is $\frac{y}{\sqrt{36 - y^2}}$.

Since gradient is negative from the diagram/context, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{\sqrt{36 - y^2}}$

(b) $w^2 = 36 - y^2$

$$2w\frac{\mathrm{d}w}{\mathrm{d}x} = -2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{w}{y}\frac{\mathrm{d}w}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{\sqrt{36 - y^2}}$$

$$-\frac{w}{y}\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{-y}{\sqrt{w^2}}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{y^2}{w\sqrt{w^2}}$$

$$= \frac{y^2}{w|w|} \qquad \left(\sqrt{w^2} = |w|\right)$$

$$= \frac{y^2}{w^2} \qquad \left(|w| = w, \text{since } w \geq 0\right)$$

$$= \frac{36 - w^2}{w^2} \qquad \text{(Shown)}$$

(c) $\frac{dw}{dx} = \frac{36-w^2}{w^2}$

$$\frac{w^2}{36-w^2}\frac{dw}{dx} = 1$$

$$\int \frac{w^2}{36-w^2}\,dw = \int 1\,dx$$

$$\int -1 + \frac{36}{36-w^2}\,dw = x + c$$

$$\int -1\,dw + 36\int \frac{1}{6^2 - w^2}\,dw = x + c$$

$$-w + 36\cdot\frac{1}{2(6)}\ln\left|\frac{6+w}{6-w}\right| = x + c$$

$$-w + 3\ln\left|\frac{6+w}{6-w}\right| = x + c$$

$$x = -w + 3\ln\left|\frac{6+w}{6-w}\right| - c$$

Since $0 < w < 6,\ x = -w + 3\ln\left(\frac{6+w}{6-w}\right) - c$

Since $w \geq 0, w^2 = 36 - y^2 \Rightarrow w = \sqrt{36-y^2}$

$$x = -\sqrt{36-y^2} + 3\ln\left(\frac{6+\sqrt{36-y^2}}{6-\sqrt{36-y^2}}\right) - c$$

From $W_0(0,6)$: When $x = 0,\ y = 6$

$$0 = -\sqrt{36-6^2} + 3\ln\frac{6+\sqrt{36-6^2}}{6-\sqrt{36-6^2}} - c$$

$$0 = 0 + 3\ln 1 + c$$

$$c = 0$$

Final equation: $x = -\sqrt{36-y^2} + 3\ln\frac{6+\sqrt{36-y^2}}{6-\sqrt{36-y^2}}$

or

$$x = -\sqrt{36-y^2} + 3\ln\left(6+\sqrt{36-y^2}\right) - 3\ln\left(6-\sqrt{36-y^2}\right)$$

(d) As $y \to 0$, $-\sqrt{36-y^2} \to -6$

$$3\ln\left(6+\sqrt{36-y^2}\right) \to 3\ln 12$$

$$-3\ln\left(6-\sqrt{36-y^2}\right) \to -3\ln 0 \to \infty$$

Thus $x = -\sqrt{36-y^2} + 3\ln\left(6+\sqrt{36-y^2}\right) - 3\ln\left(6-\sqrt{36-y^2}\right)$ increases to infinity.

(or $\dfrac{6+\sqrt{36-y^2}}{6-\sqrt{36-y^2}}$ tends to infinity)