

1

$$(i) \quad x = 3 \sec t \Rightarrow \sec t = \frac{x}{3}$$

$$y = -2 \tan t \Rightarrow \tan t = -\frac{y}{2}$$

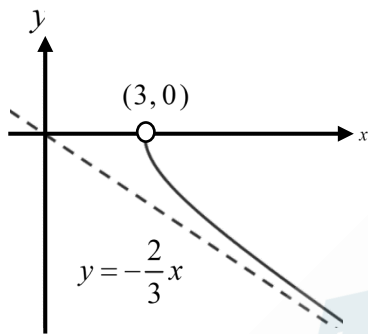
$$\tan^2 t + 1 = \sec^2 t$$

$$\left(-\frac{y}{2}\right)^2 + 1 = \left(\frac{x}{3}\right)^2$$

$$\frac{y^2}{2^2} + 1 = \frac{x^2}{3^2}$$

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1, \quad x > 3, y < 0$$

(ii)



2

$$\begin{aligned} \text{(a)} \quad \frac{dx}{dt} &= -\frac{1}{t^2}, \quad \frac{dy}{dt} = -\frac{8}{t^3} - 1 \\ \frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} \\ &= \left( -\frac{8}{t^3} - 1 \right) (-t^2) \\ &= \frac{8}{t} + t^2 \end{aligned}$$

For stationary point,

$$\begin{aligned} \frac{dy}{dx} = 0 &\Rightarrow \frac{8}{t} + t^2 = 0 \\ \Rightarrow t^3 &= -8 \\ \Rightarrow t &= -2 \end{aligned}$$

$$\text{When } t = -2, \quad x = \frac{1}{-2} + 2 = \frac{3}{2}, \quad y = \frac{4}{(-2)^2} + 2 = 3$$

Coordinates of  $A$  is  $\left(\frac{3}{2}, 3\right)$

Equation of tangent is  $y - 3 = 0 \left(x - \frac{3}{2}\right)$   $\widehat{ABC}$

$$y = 3$$

$$\text{(b)} \quad \text{When } y = 3, \quad \frac{4}{t^2} - t = 3$$

$$\begin{aligned} \Rightarrow 4 - t^3 &= 3t^2 \\ \Rightarrow t^3 + 3t^2 - 4 &= 0 \end{aligned}$$

Using GC,  $t = 1$  or  $t = -2$  (Reject  $\because t = -2$  is pt  $A$ )

[If GC is not allowed, note that  $t = -2$  must be one of the solution since one of the intersection point between the tangent and  $C$  is at  $A$

$$\Rightarrow (t+2)(t^2+t-2) = 0$$

$$\Rightarrow (t+2)^2(t-1) = 0$$

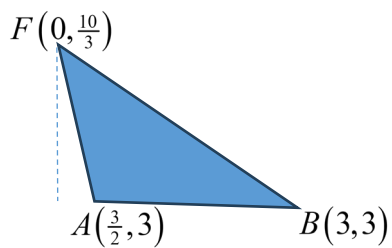
$$\Rightarrow t = -2 \text{ or } t = 1 \quad ]$$

$\therefore$  coordinates of  $B$  is  $(3, 3)$ .

$$\text{Gradient of normal at } B = -\frac{1}{\frac{8}{1} + 1^2} = -\frac{1}{9}$$

$$\text{Equation of normal is } y - 3 = -\frac{1}{9}(x - 3), \text{ i.e. } y = -\frac{1}{9}x + \frac{10}{3}$$

- (c) The point  $F$  has coordinates  $\left(0, \frac{10}{3}\right)$ .



$$\text{Height of triangle} = \frac{10}{3} - 3 = \frac{1}{3}$$

$$AB \text{ has length } 3 - \frac{3}{2} = \frac{3}{2}$$

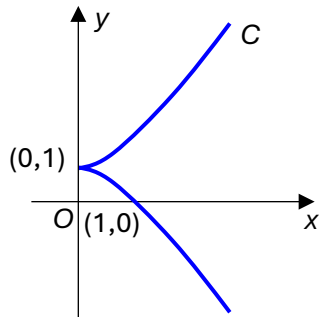
$$\text{Area of Triangle} = \frac{1}{2} \left(\frac{1}{3}\right) \left(\frac{3}{2}\right) = \frac{1}{4} \text{ unit}^2$$



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- (a) When  $x = 0$ ,  $t = 0$  and  $y = 1$   
When  $y = 0$ ,  $t = -1$  and  $x = 1$



(b)  $\frac{dx}{dt} = 2t$ ,  $\frac{dy}{dt} = 3t^2$   
 $\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3}{2}t$

When  $t = 1$ ,  $x = 1$ ,  $y = 2$  and  $\frac{dy}{dx} = \frac{3}{2}$

$\therefore$  equation of tangent at  $P$  is

$$y - 2 = \frac{3}{2}(x - 1)$$

$$y = \frac{3}{2}x + \frac{1}{2} \quad \dots(*)$$

- (c) Substitute  $x = t^2$  and  $y = t^3 + 1$  into (\*):

$$t^3 + 1 = \frac{3}{2}t^2 + \frac{1}{2}$$

$$2t^3 + 2 = 3t^2 + 1$$

$$2t^3 - 3t^2 + 1 = 0$$

$$(t - 1)(2t^2 - t - 1) = 0$$

$$(t - 1)(2t + 1)(t - 1) = 0$$

$\therefore t = -\frac{1}{2}$  or  $t = 1$  (reject since this is  $P$ )

When  $t = -\frac{1}{2}$ ,  $x = \frac{1}{4}$  and  $y = \frac{7}{8}$

Hence coordinates of  $Q$  is  $\left(\frac{1}{4}, \frac{7}{8}\right)$ .

4

(a)  $\frac{dy}{dt} = \cos t$  and  $\frac{dx}{dt} = -2 \sin 2t$

When  $t = \frac{\pi}{3}$ ,

$$\frac{dy}{dx} = \frac{\cos t}{-2 \sin 2t} = \frac{\frac{1}{2}}{-2 \left( \frac{\sqrt{3}}{2} \right)} = -\frac{1}{2\sqrt{3}}$$

$$x = -1 + \cos 2 \left( \frac{\pi}{3} \right) = -1 + \cos \left( \frac{2\pi}{3} \right)$$

$$y = 2 + \sin \frac{\pi}{3}$$

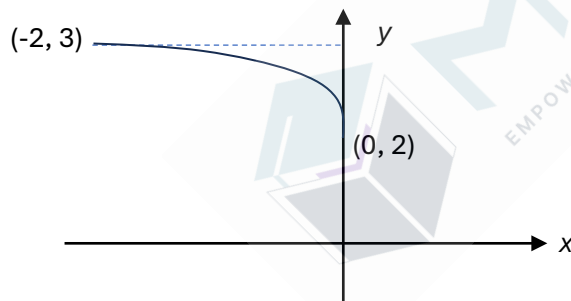
Grad of normal at  $t = \frac{\pi}{3}$ :  $2\sqrt{3}$

$$y - \left( 2 + \sin \frac{\pi}{3} \right) = 2\sqrt{3} \left( x - \left( -1 + \cos \frac{2\pi}{3} \right) \right)$$

$$y - 2 - \frac{\sqrt{3}}{2} = 2\sqrt{3}x + 3\sqrt{3}$$

$$y = 2\sqrt{3}x + \frac{4 + 7\sqrt{3}}{2}$$

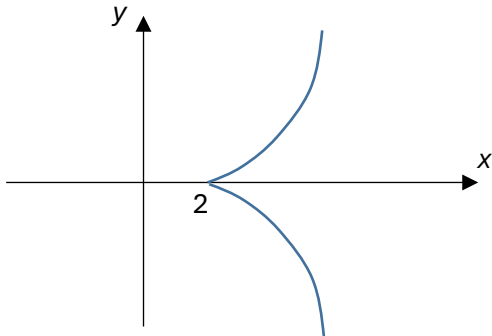
(b)



When  $t = 0$ ,  $x = 0$ ,  $\frac{dy}{dx}$  is undefined.

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(a)



(b) At  $(6, 8)$ ,

$$6 = t^2 + 2$$

$$t^2 = 4$$

$$t = \pm 2$$

$$\text{When } t = 2, y = 2^3 = 8$$

$$\text{When } t = -2, y = 2^3 = -8$$

$$\therefore t = 2$$

$$\frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = 3t^2 \times \frac{1}{2t} = \frac{3}{2}t$$

$$\text{When } t = 2, \frac{dy}{dx} = 3$$

$$\text{Equation of tangent is } y - 8 = 3(x - 6)$$

$$\text{i.e., } y = 3x - 10$$

(c) Since tangent to  $C$  at the point  $(6, 8)$  meets the curve  $C$  again at point  $P$ ,  $t^3 = 3(t^2 + 2) - 10$

$$t^3 - 3t^2 + 4 = 0$$

Using GC,

$$t = 2 \quad \text{or} \quad t = -1$$

(Point  $(6, 8)$ )

**Alt**

$$(t - 2)(t^2 - t - 2) = 0$$

$$(t - 2)(t - 2)(t + 1) = 0$$

$$t = 2 \quad \text{or} \quad t = -1$$

At  $t = -1$ ,  $x = 3$  and  $y = -1$ .

The coordinates of point  $P$  are  $(3, -1)$ .

(d) At  $t = m$ , the normal to the curve is

$$y - m^3 = -\frac{2}{3m}(x - m^2 - 2)$$

$$\text{i.e., } y = -\frac{2}{3m}x + \frac{2m}{3} + \frac{4}{3m} + m^3$$

$$\text{When } x = 0, y = \frac{2m}{3} + \frac{4}{3m} + m^3 \text{ (Point } R)$$

$$\text{When } y = 0, x = \frac{3m^4}{2} + m^2 + 2 \text{ (Point } Q)$$

$$\text{The mid-point } F \text{ is } \left( \frac{3m^4}{4} + \frac{m^2}{2} + 1, \frac{m}{3} + \frac{2}{3m} + \frac{m^3}{2} \right).$$



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(a)  $x = \sqrt{3} \sin 2t \Rightarrow \frac{dx}{dt} = 2\sqrt{3} \cos 2t$

$$y = 4 \cos^2 t \Rightarrow \frac{dy}{dt} = 8 \cos t (-\sin t) = -4 \sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{-4 \sin 2t}{2\sqrt{3} \cos 2t}$$

$$= \frac{-2}{\sqrt{3}} \tan 2t$$

$$= -\frac{2\sqrt{3}}{3} \tan 2t \equiv k\sqrt{3} \tan 2t$$

where  $k = -\frac{2}{3}$ .

(b) When  $t = \frac{\pi}{4}$ ,  $x = \sqrt{3} \sin 2\left(\frac{\pi}{4}\right) = \sqrt{3}$ ,  $y = 4 \cos^2\left(\frac{\pi}{4}\right) = 2$ ,

$$\frac{dy}{dx} = -\frac{2\sqrt{3}}{3} \tan \frac{\pi}{2}, \text{ which is undefined.}$$

$\Rightarrow$  The tangent is parallel to the  $y$ -axis.

Hence the equation of the tangent at the point where  $t = \frac{\pi}{4}$

is  $x = \sqrt{3}$ .

When  $t = \frac{\pi}{3}$ ,  $x = \sqrt{3} \sin 2\left(\frac{\pi}{3}\right) = \sqrt{3} \left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$ ,

$$y = 4 \cos^2\left(\frac{\pi}{3}\right) = 1$$

$$\frac{dy}{dx} = -\frac{2\sqrt{3}}{3} \tan \frac{2\pi}{3} = -\frac{2\sqrt{3}}{3} (-\sqrt{3}) = 2$$

OR

From GC, when  $t = \frac{\pi}{3}$ ,  $x = \frac{3}{2}$ ,  $y = 1$ ,  $\frac{dy}{dx} = 2$ .

Equation of the tangent is:

$$y - 1 = 2\left(x - \frac{3}{2}\right)$$

$$y = 2x - 2$$

(c) Let  $\theta$  be the angle in which the tangent  $y = 2x - 2$  makes with the positive  $x$ -axis.

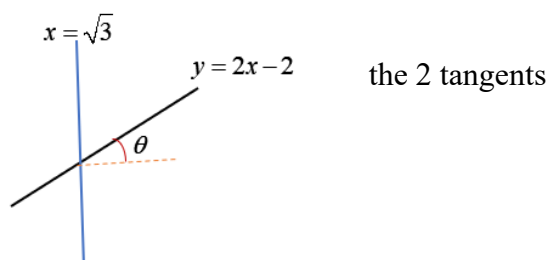
Then  $\tan \theta = 2$ .

Hence, acute angle between

$$= 90^\circ - \theta$$

$$= 90^\circ - \tan^{-1} 2$$

$$\approx 26.6^\circ$$



7

(i)

$$x = \sqrt{4+t^2} \quad y = t^2$$
$$\frac{dx}{dt} = \frac{t}{\sqrt{4+t^2}} \quad \frac{dy}{dt} = 2t$$
$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= 2\sqrt{4+t^2}$$

When  $t = p$ ,  $x = \sqrt{4+p^2}$ ,  $y = p^2$  and  $\frac{dy}{dx} = 2\sqrt{4+p^2}$ .

Equation of tangent

$$y - p^2 = 2\sqrt{4+p^2} (x - \sqrt{4+p^2})$$

$$y = 2\sqrt{4+p^2}x - 8 - p^2$$

(ii)

When  $y = 0$ ,  $x = \frac{8+p^2}{2\sqrt{4+p^2}}$

$x$ -coordinate of  $M$

$$= \frac{1}{2} \left( \sqrt{4+p^2} + \frac{8+p^2}{2\sqrt{4+p^2}} \right)$$

$$= \frac{16+3p^2}{4\sqrt{4+p^2}}$$

Coordinates of  $M$  are

$$\left( \frac{16+3p^2}{4\sqrt{4+p^2}}, \frac{p^2}{2} \right)$$

Parametric equation of curve traced out by  $M$

$$x = \frac{16+3p^2}{4\sqrt{4+p^2}}$$

$$y = \frac{p^2}{2} \quad (1)$$

From (1),  $p^2 = 2y$

Cartesian equation of line

$$x = \frac{16+3(2y)}{4\sqrt{4+2y}}$$

$$2x\sqrt{4+2y} = 8+3y$$

8

(a) When  $x = 0$ ,  $3t^3 + 2t^2 = 0$ .

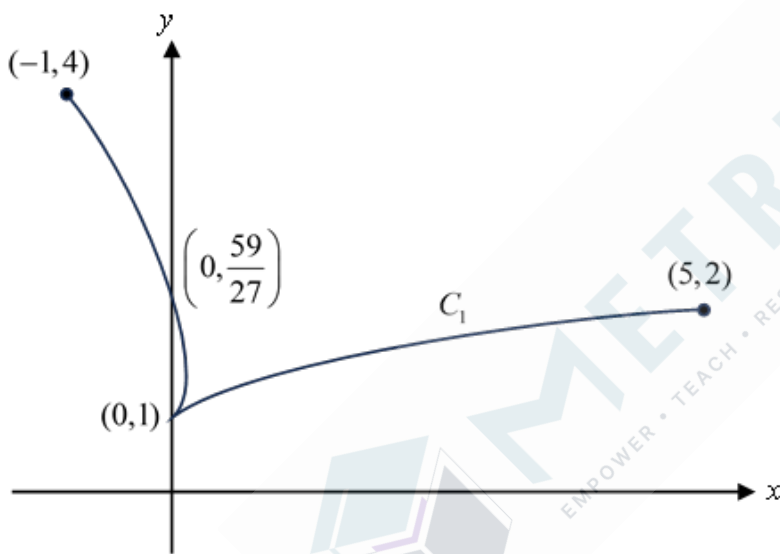
$$t^2(3t+2)=0, t=0 \text{ or } t=-\frac{2}{3}$$

When  $t = 0$ ,  $y = 0^2 + 1 = 1$ .

$$\text{When } t = -\frac{2}{3}, y = -\left(-\frac{2}{3}\right)^3 + 2\left(-\frac{2}{3}\right)^2 + 1 = \frac{59}{27}.$$

When  $t = -1$ ,  $x = -1$  and  $y = 4$ .

When  $t = 1$ ,  $x = 5$  and  $y = 2$ .



(b) The curve is a circle with centre  $(0, 1)$  and radius  $\sqrt{k}$ .

(c) Substitute  $x = 3t^3 + 2t^2$  and  $y = -t^3 + 2t^2 + 1$  into  $C_2$ ,

$$(3t^3 + 2t^2)^2 + (-t^3 + 2t^2 + 1 - 1)^2 = 16$$

$$(9t^6 + 12t^5 + 4t^4) + (t^6 - 4t^5 + 4t^4) - 16 = 0$$

$$10t^6 + 8t^5 + 8t^4 - 16 = 0$$

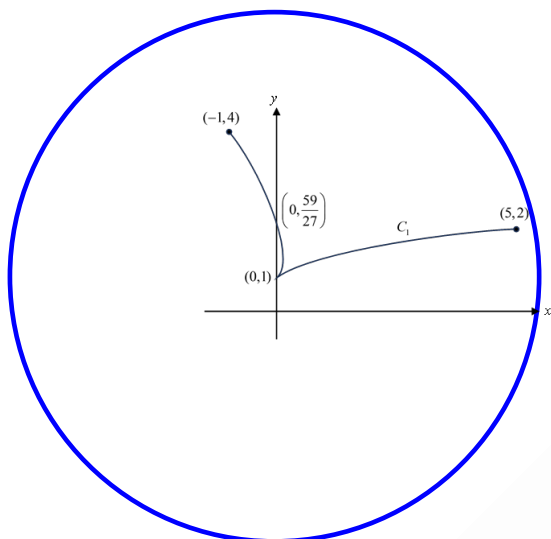
Using GC,

$t = 0.90823$  (reject root smaller than  $-1$  and non-real roots)

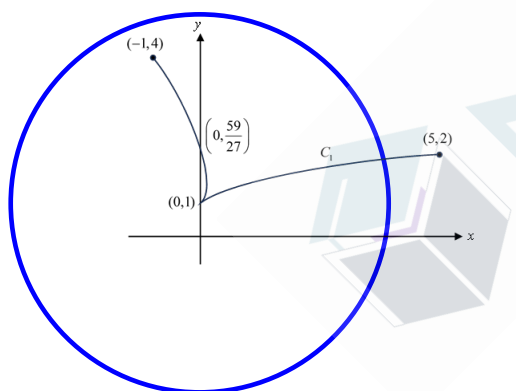
Substituting,  $x = 3.8973$  and  $y = 1.9006$ .

The coordinates of the point of intersection are  $(3.90, 1.90)$ .

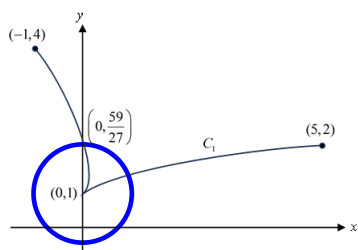
(d) Circle with centre  $(0, 1)$  and radius  $\sqrt{k}$ .



0 point of intersection



1 point of intersection



2 points of intersection

Hence, there can be 0, 1 or 2 points of intersection.

9

$$\sec \theta = \frac{x-1}{3}$$

$$\tan \theta = \frac{y-3}{2}$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{y-3}{2}\right)^2 = 1$$

$$P: (1,3)$$

$$Q: (-2,3)$$

$$R: (4,3)$$

Let  $y = 0$

$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{0-3}{2}\right)^2 = 1$$

$$\left(\frac{x-1}{3}\right)^2 = 1 + \left(\frac{-3}{2}\right)^2$$

$$(x-1)^2 = 9\left(1 + \frac{9}{4}\right)$$

$$x = 1 \pm \frac{3\sqrt{13}}{2}$$

$$S: \left(1 - \frac{3\sqrt{13}}{2}, 0\right)$$

To find asymptotes:

$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{y-3}{2}\right)^2 = 0$$

$$\left(\frac{y-3}{2}\right)^2 = \left(\frac{x-1}{3}\right)^2$$

$$\frac{y-3}{2} = \pm \left(\frac{x-1}{3}\right)$$

$$y-3 = \pm 2\left(\frac{x-1}{3}\right)$$

$$y = 3 + 2\left(\frac{x-1}{3}\right) \quad \text{or} \quad y = 3 - 2\left(\frac{x-1}{3}\right)$$

$$y = \frac{2}{3}x + \frac{7}{3}$$

$$y = -\frac{2}{3}x + \frac{11}{3}$$

The asymptotes are  $y = \frac{2}{3}x + \frac{7}{3}$  and  $y = -\frac{2}{3}x + \frac{11}{3}$ .



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$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{y-3}{2}\right)^2 = 1$$

↓ Replace  $y$  with  $y+3$

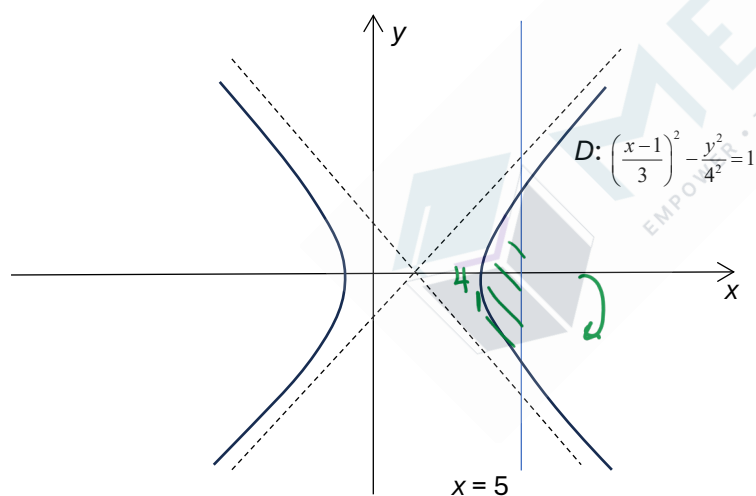
$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{y}{2}\right)^2 = 1$$

↓ Replace  $y$  with  $\frac{y}{2}$

$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{\frac{y}{2}}{2}\right)^2 = 1 \quad (\text{i.e. } \left(\frac{x-1}{2}\right)^2 - \left(\frac{y}{4}\right)^2 = 1)$$

Translate  $C$  by 3 units in the negative  $y$ -direction, followed by scale parallel to the  $y$ -axis by factor 2.

$$\begin{aligned} \text{Volume} &= \pi \int_4^5 \frac{16}{9}(x-1)^2 - 16 \, dx \\ &= \pi \left[ \frac{16(x-1)^3}{9(3)} - 16x \right]_4^5 \\ &= \frac{160\pi}{27} \text{ or } 18.6 \text{ (by GC)} \end{aligned}$$



10

(a) When  $y = 0$ ,

$$ut \sin \theta - \frac{1}{2}gt^2 = 0$$

$$t \left( u \sin \theta - \frac{1}{2}gt \right) = 0$$

$$t = 0 \quad \text{or} \quad t = \frac{2u \sin \theta}{g}$$

Since  $t \neq 0$ ,  $t = \frac{2u \sin \theta}{g}$ .

$t = \frac{2u \sin \theta}{g}$  represents the time taken for the projectile to return back to the same height as the origin at which it is launched.

(b) When  $y = 0$ ,  $t = \frac{2u \sin \theta}{g}$ ,

$$x = u \left( \frac{2u \sin \theta}{g} \right) \cos \theta$$

$$= \frac{u^2}{g} \sin 2\theta \quad (\text{Shown})$$

(c)  $\frac{dx}{d\theta} = \frac{2u^2}{g} \cos 2\theta$

When  $\frac{dx}{d\theta} = 0$ ,  $\frac{2u^2}{g} \cos 2\theta = 0$

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4} \quad \left( \text{since } 0 < \theta < \frac{\pi}{2} \right)$$

$$\frac{d^2x}{d\theta^2} = -\frac{4u^2}{g} \sin 2\theta$$

When  $\theta = \frac{\pi}{4}$ ,  $\frac{d^2x}{d\theta^2} = -\frac{4u^2}{g} < 0$

$\theta = \frac{\pi}{4}$  gives the maximum range of the projectile.

$$x = \frac{u^2}{g} \sin \left( 2 \left( \frac{\pi}{4} \right) \right)$$

$$= \frac{u^2}{g}$$

Hence the maximum range of the projectile is  $\frac{u^2}{g}$ .