1

(a)

$$\sqrt{4-x}=(4-x)^{\frac{1}{2}}$$
$$=\left[4\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}$$
$$=4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$
$$=2\left(1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right)$$
$$=2\left(1-\frac{x}{8}-\frac{x^2}{128}+\ldots\right)$$
$$=2-\frac{x}{4}-\frac{x^2}{64}+\ldots$$

(b)

$$\theta\left(\sqrt{3}+\sin 2\theta\right)-2\cos\left(\frac{\pi}{3}-\theta\right)$$
$$\approx\theta\left(\sqrt{3}+2\theta\right)-2\left(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta\right)$$
$$\approx\sqrt{3}\theta+2\theta^2-2\left(\frac{1}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{\sqrt{3}}{2}\theta\right)$$
$$=\sqrt{3}\theta+2\theta^2-1+\frac{\theta^2}{2}-\sqrt{3}\theta$$
$$=\frac{5\theta^2}{2}-1$$

OR

$$\theta\left(\sqrt{3}+\sin 2\theta\right)-2\cos\left(\frac{\pi}{3}-\theta\right)$$
$$=\theta\left(\sqrt{3}+2\sin\theta\cos\theta\right)-2\left(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta\right)$$
$$\approx\sqrt{3}\theta+2(\theta)(\theta)\left(1-\frac{\theta^2}{2}\right)-2\left(\frac{1}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{\sqrt{3}}{2}\theta\right)$$
$$\approx\sqrt{3}\theta+2\theta^2-1+\frac{\theta^2}{2}-\sqrt{3}\theta$$
$$=\frac{5\theta^2}{2}-1$$

2

By sine rule, $\dfrac{PQ}{\sin(\frac{\pi}{3}-\theta)} = \dfrac{PR}{\sin\frac{\pi}{3}}$

$$\therefore \frac{PR}{PQ} = \frac{\sin\frac{\pi}{3}}{\sin(\frac{\pi}{3}-\theta)}$$

$$= \frac{\sin\frac{\pi}{3}}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$$

$$\approx \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}(1-\frac{\theta^2}{2}) - \frac{1}{2}\theta}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}[(1-\frac{\theta^2}{2}) - \frac{1}{\sqrt{3}}\theta]}$$

$$= \frac{1}{1-\frac{1}{\sqrt{3}}\theta - \frac{1}{2}\theta^2}$$

$$= \left[1-(\tfrac{1}{\sqrt{3}}\theta + \tfrac{1}{2}\theta^2)\right]^{-1}$$

$$\approx 1 + (\tfrac{1}{\sqrt{3}}\theta + \tfrac{1}{2}\theta^2) + (\tfrac{1}{\sqrt{3}}\theta + \tfrac{1}{2}\theta^2)^2$$

$$\approx 1 + \tfrac{1}{\sqrt{3}}\theta + \tfrac{1}{2}\theta^2 + \tfrac{1}{3}\theta^2$$

$$= 1 + \tfrac{1}{\sqrt{3}}\theta + \tfrac{5}{6}\theta^2 \quad \text{where } \alpha = \tfrac{1}{\sqrt{3}},\ \beta = \tfrac{5}{6} \quad \text{(shown)}$$

3

(a)

$$y = e^{-x}\sin x + x - 1$$

$$\frac{dy}{dx} = e^{-x}\cos x - e^{-x}\sin x + 1$$

$$= e^{-x}(\cos x - \sin x) + 1$$

$$\frac{d^2y}{dx^2} = e^{-x}(-\sin x - \cos x) - e^{-x}(\cos x - \sin x)$$

$$= -2e^{-x}\cos x \quad \text{where } k = -2 \text{ (shown)}$$

(b)

$$\frac{d^3y}{dx^3} = -2e^{-x}(-\sin x) - (-2e^{-x})\cos x$$

$$= 2e^{-x}(\sin x + \cos x)$$

When $x = 0: f(0) = -1$

$f'(0) = 2$

$f''(0) = -2$

$f'''(0) = 2$

$$y = -1 + 2x - 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^3}{3!} + \ldots$$

$$= -1 + 2x - x^2 + \frac{x^3}{3} + \ldots$$

(c)

$$\frac{e^{-x}\sin x + x - 1}{\cos 2x} = \frac{-1 + 2x - x^2 + \frac{x^3}{3} + \ldots}{1 - \frac{(2x)^2}{2!} + \ldots}$$

$$= \left(-1 + 2x - x^2 + \frac{x^3}{3} + \ldots\right)(1 - 2x^2)^{-1}$$

$$= \left(-1 + 2x - x^2 + \frac{x^3}{3} + \ldots\right)\left[1 + (-1)(-2x^2) + \ldots\right]$$

$$= \left(-1 + 2x - x^2 + \frac{x^3}{3} + \ldots\right)(1 + 2x^2 + \ldots)$$

$$= -1 - 2x^2 + 2x + 4x^3 - x^2 + \frac{x^3}{3} + \ldots$$

$$= -1 + 2x - 3x^2 + \frac{13}{3}x^3 + \ldots$$

4

(a)

$$f(x) = \sec^2 x$$

$$f'(x) = 2\sec x(\sec x \tan x)$$

$$= 2\sec^2 x \tan x$$

$$f''(x) = 2\left[\sec^2 x(\sec^2 x) + (\tan x)(2\sec x(\sec x \tan x))\right]$$

$$= 2\sec^4 x + 4\sec^2 x \tan^2 x$$

When $x = 0$, $f(0) = 1$

$$f'(x) = 0$$

$$f''(0) = 2$$

$$\therefore f(x) = 1 + 0x + \frac{2}{2!}x^2 + \ldots$$

$$= 1 + x^2 + \ldots$$

(b)

$$(1+ax)^{-4} = 1 + (-4)(ax) + \frac{(-4)(-5)}{2!}(ax)^2 + \ldots$$

$$= 1 - 4ax + 10a^2x^2 + \ldots$$

Since the coefficients of the x and x^2 terms in the expansion are equal,

$$-4a = 10a^2$$

$$10a^2 + 4a = 0$$

$$2a(5a+2) = 0$$

$a = 0$ (rejected since $a \neq 0$) or $a = -\frac{2}{5}$

5

(a)

$$\begin{aligned}
g(x) &= \frac{x}{\sqrt{4-x}} \\
&= x(4-x)^{-\frac{1}{2}} \\
&= x\left[4\left(1-\frac{x}{4}\right)\right]^{-\frac{1}{2}} \\
&= 4^{-\frac{1}{2}}x\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2}x\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2}x\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\cdots\right) \\
&= \frac{1}{2}x+\frac{1}{16}x^2+\frac{3}{256}x^3+\cdots
\end{aligned}$$

Therefore $a=\frac{1}{2}$, $b=\frac{1}{16}$, $c=\frac{3}{256}$

(b)

Percentage error $=\left|\frac{f(x)-g(x)}{g(x)}\right|\times 100\% < 4\%$

$$\Rightarrow \left|\frac{\left(\frac{1}{2}x+\frac{1}{16}x^2+\frac{3}{256}x^3\right)-\frac{x}{\sqrt{4-x}}}{\frac{x}{\sqrt{4-x}}}\right| < 0.04$$

Using GC, $\Rightarrow$ $0 < x < 1.87$ (corr. to 3 s.f.)

6

(a)

$$f(x)=\frac{1}{\sqrt{4-x^2}}=\left(4-x^2\right)^{-\frac{1}{2}}=\left[4\left(1-\frac{x^2}{4}\right)\right]^{-\frac{1}{2}}$$

$$=\frac{1}{2}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x^2}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{x^2}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(-\frac{x^2}{4}\right)^3+\ldots\right]$$

$$=\frac{1}{2}\left[1+\frac{x^2}{8}+\frac{3x^4}{128}+\frac{5x^6}{1024}+\ldots\right]$$

$$=\frac{1}{2}+\frac{x^2}{16}+\frac{3x^4}{256}+\frac{5x^6}{2048}+\ldots$$

(b) **Method 1**

$$\frac{\sec 2x}{\sqrt{4-x^2}}=\frac{1}{\sqrt{4-x^2}}\cdot\frac{1}{\cos 2x}$$

$$=\left(\frac{1}{2}+\frac{x^2}{16}+\frac{3x^4}{256}+\frac{5x^6}{2048}+\ldots\right)\left(\frac{1}{1-\frac{(2x)^2}{2}+\ldots}\right)$$

$$=\left(\frac{1}{2}+\frac{x^2}{16}+\ldots\right)\left(1-2x^2+\ldots\right)^{-1}$$

$$=\left(\frac{1}{2}+\frac{x^2}{16}+\ldots\right)\left(1+2x^2+\ldots\right)$$

$$=\frac{1}{2}+\frac{17x^2}{16}+\ldots$$

Method 2

$y=\sec 2x$

$$\frac{dy}{dx}=2\sec 2x\tan 2x$$

$$\frac{d^2y}{dx^2}=2\left[2\sec 2x\tan^2 2x+2\sec^3 2x\right]$$

When $x=0$, $y=1,\frac{dy}{dx}=0,\frac{d^2y}{dx^2}=4$

$$\sec 2x=1+\frac{4}{2!}x^2+\ldots=1+2x^2+\ldots$$

$$\frac{\sec 2x}{\sqrt{4-x^2}}=\left(\frac{1}{2}+\frac{x^2}{16}+\frac{3x^4}{256}+\frac{5x^6}{2048}+\ldots\right)\left(1+2x^2+\ldots\right)=\frac{1}{2}+\frac{17x^2}{16}+\ldots$$

7

(i) **Method 1**

$$\ln\left(1+e^{-x}\right) \approx \ln\left[1+\left(1-x+\frac{x^2}{2}\right)\right]$$

$$= \ln\left(2-x+\frac{x^2}{2}\right)$$

$$= \ln\left[2\left(1-\frac{x}{2}+\frac{x^2}{4}\right)\right]$$

$$= \ln 2 + \ln\left[1+\left(-\frac{x}{2}+\frac{x^2}{4}\right)\right]$$

$$= \ln 2 + \left(-\frac{x}{2}+\frac{x^2}{4}\right) - \frac{\left(-\frac{x}{2}+\frac{x^2}{4}\right)^2}{2} + \dots$$

$$= \ln 2 - \frac{x}{2} + \frac{x^2}{4} - \left(\frac{x^2}{8}\right) + \dots$$

$$= \ln 2 - \frac{x}{2} + \frac{1}{8}x^2 + \dots$$

Method 2

$$f(x) = \ln\left(1+e^{-x}\right)$$

$$f'(x) = \frac{1}{1+e^{-x}}\left(-e^{-x}\right) = -\frac{1}{e^x+1}$$

$$f''(x) = \left(e^x+1\right)^{-2} e^x$$

When $x=0$, $f(0)=\ln 2$, $f'(0) = -\frac{1}{2}$, $f''(0) = \frac{1}{4}$

$$f(x) = \ln 2 - \frac{1}{2}x + \left(\frac{1}{4}\right)\frac{x^2}{2!} + \dots\dots$$

$$f(x) = \ln 2 - \frac{1}{2}x + \frac{x^2}{8} + \dots\dots$$

(ii) $$\frac{d}{dx}\ln\left(1+e^{-x}\right) = \frac{-e^{-x}}{1+e^{-x}} = -\frac{1}{1+e^x}$$

Using the series from part **(i)**, $\frac{1}{1+e^x} \approx -\frac{d}{dx}\left(\ln 2 - \frac{x}{2} + \frac{1}{8}x^2\right) = -\left(-\frac{1}{2}+\frac{2}{8}x\right) = \frac{1}{2} - \frac{1}{4}x$.

8

(a) $y = e^x \cos 3x$

$$\frac{dy}{dx} = e^x \cos 3x + e^x(-3\sin 3x)$$
$$= y - 3e^x \sin 3x$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} - 3\left[e^x \sin 3x + e^x\left(3\cos 3x\right)\right]$$
$$= \frac{dy}{dx} - 3e^x \sin 3x - 3e^x\left(3\cos 3x\right)$$
$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - 9y$$
$$= 2\frac{dy}{dx} - 10y \text{ (shown)}$$

$$\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2} - 10\frac{dy}{dx}$$

When $x = 0$,

$$y = 1,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = -8,\ \frac{d^3y}{dx^3} = -26$$

By Maclaurin expansion,

$$y = 1 + x + \frac{(-8)}{2!}x^2 + \frac{(-26)}{3!}x^3 + \ldots$$
$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

(b) Using standard series expansion,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$\cos 3x = 1 - \frac{(3x)^2}{2!} + \ldots = 1 - \frac{9}{2}x^2 + \ldots$$

$$e^x \cos 3x = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots\right)\left(1 - \frac{9}{2}x^2 + \ldots\right)$$
$$= 1 + x + \left(\frac{1}{2} - \frac{9}{2}\right)x^2 + \left(\frac{1}{6} - \frac{9}{2}\right)x^3 + \ldots$$
$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots \text{ (verified)}$$

(c)

$$\ln(1+e^{x}\cos 3x)$$

$$\approx \ln\left[1+\left(1+x-4x^{2}\right)\right]$$

$$= \ln\left(2+x-4x^{2}\right)$$

$$= \ln 2\left(1+\frac{x}{2}-2x^{2}\right)$$

$$= \ln 2+\ln\left(1+\frac{x}{2}-2x^{2}\right)$$

$$= \ln 2+\left(\frac{x}{2}-2x^{2}\right)-\frac{\left(\frac{x}{2}-2x^{2}\right)^{2}}{2}+\dots$$

$$= \ln 2+\frac{x}{2}-2x^{2}-\frac{1}{2}\left(\frac{x^{2}}{4}\right)+\dots$$

$$= \ln 2+\frac{1}{2}x-\frac{17}{8}x^{2}+\dots \text{ (shown)}$$

9

(a)

$$y = \frac{2\cos 2x}{1+\sin 2x}$$

$$\approx 2\left(1-\frac{(2x)^2}{2!}\right)(1+2x)^{-1}$$

$$\approx 2\left(1-2x^2\right)\left(1+(-1)2x+\frac{(-1)(-2)}{2!}(2x)^2\right)$$

$$= 2\left(1-2x^2\right)\left(1-2x+4x^2\right)$$

$$\approx 2\left(1-2x+4x^2-2x^2\right) = 2-4x+4x^2$$

(b)(i)

$$y = \ln\left(1+\sin 2x\right)$$

$$e^y = 1+\sin 2x$$

Differentiate wrt x

$$e^y \frac{dy}{dx} = 2\cos 2x \;---(1)$$

Differentiate wrt x

$$e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 = -4\sin 2x$$

$$e^y\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right) = -4\sin 2x \;---(2)$$

$h = 2, k = -4$

(b)(ii) Differentiate wrt x

$$e^y\left(\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\right)+e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right) = -8\cos 2x \;\;---(3)$$

When $x = 0,\; y = 0$

From (1) : $e^0 \frac{dy}{dx} = 2\cos 0 \Rightarrow \frac{dy}{dx} = 2$

From (2) : $e^0\left(\frac{d^2y}{dx^2}+(2)^2\right) = 0 \Rightarrow \frac{d^2y}{dx^2} = -4$

From (3) : $e^0\left(\frac{d^3y}{dx^3}+2(2)(-4)\right)+e^0(2)\left(-4+(2)^2\right) = -8 \Rightarrow \frac{d^3y}{dx^3} = 8$

$$y = 2x+\frac{(-4)}{2}x^2+\frac{(8)}{3!}x^3+\ldots = 2x-2x^2+\frac{4}{3}x^3+\ldots$$

(c) $\ln(1+\sin 2x) \approx 2x - 2x^2 + \frac{4}{3}x^3 + \ldots$

differentiating both sides,

$$\frac{d}{dx}\ln(1+\sin 2x) = \frac{d}{dx}\left(2x - 2x^2 + \frac{4}{3}x^3 + \ldots\right)$$

$$\frac{2\cos 2x}{1+\sin 2x} = 2 - 4x + 4x^2$$

10

(a)(i) $y=\ln\left(1+2x+3x^2\right)$

$e^y=1+2x+3x^2$

Differentiating with respect to x,

$$\frac{dy}{dx}e^y=2+6x$$

$$\frac{d^2y}{dx^2}e^y+\left(\frac{dy}{dx}\right)^2e^y=6$$

When $x=0$,

$y=0$

$\frac{dy}{dx}=2$

$\frac{d^2y}{dx^2}=2$

$$y=0+(2)x+\frac{2}{2!}x^2+\ldots$$

$$=2x+x^2+\ldots\quad\left(\text{up to } x^2\right)$$

(ii) $y=\ln\left(1+2x+3x^2\right)$

$$=2x+3x^2-\frac{\left(2x+3x^2\right)^2}{2}+\ldots\quad(\text{using standard series})$$

$$=2x+3x^2-\frac{(2x)^2}{2}+\ldots$$

$$=2x+x^2+\ldots\quad\left(\text{up to } x^2\right)(\text{verified})$$

(b) $\frac{x}{\sqrt{4+x}}=x(4+x)^{-\frac{1}{2}}$

$$=x\left[4\left(1+\frac{x}{2}\right)\right]^{-\frac{1}{2}}$$

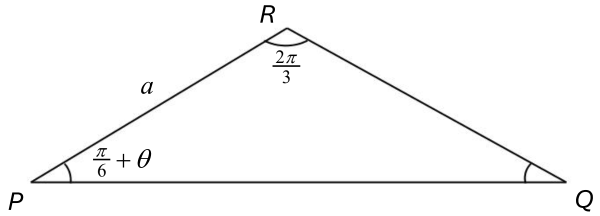
$$=x(4)^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$$

$$=\frac{x}{2}\left(1+\left(-\frac{1}{2}\right)\frac{x}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{x}{4}\right)^2+\ldots\right)$$

$$=\frac{1}{2}x-\frac{1}{16}x^2+\frac{3}{256}x^3\quad\left(\text{up to } x^3\right)$$

11

(i)



$$\angle PQR = \pi - \frac{2\pi}{3} - \left(\frac{\pi}{6} + \theta\right)$$

$$= \frac{\pi}{6} - \theta$$

Using Sine Rule,

$$\frac{PQ}{\sin\frac{2\pi}{3}} = \frac{a}{\sin\left(\frac{\pi}{6} - \theta\right)}$$

$$PQ = \frac{a\sin\frac{2\pi}{3}}{\sin\frac{\pi}{6}\cos\theta - \cos\frac{\pi}{6}\sin\theta}$$

$$= \frac{a\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta}$$

$$= \frac{\sqrt{3}\,a}{\cos\theta - \sqrt{3}\sin\theta}$$

(ii)

$$PQ = \frac{\sqrt{3}\,a}{\cos\theta - \sqrt{3}\sin\theta}$$

$$\approx \frac{\sqrt{3}\,a}{1 - \frac{\theta^2}{2} - \sqrt{3}\,\theta}$$

$$= \sqrt{3}\,a\left(1 - \sqrt{3}\,\theta - \frac{\theta^2}{2}\right)^{-1}$$

$$= \sqrt{3}\,a\left(1 + (-1)\left(-\sqrt{3}\,\theta - \frac{\theta^2}{2}\right) + \frac{(-1)(-1)}{2!}\left(-\sqrt{3}\,\theta - \frac{\theta^2}{2}\right)^2 + \cdots\right)$$

$$= \sqrt{3}\,a\left(1 + \left(\sqrt{3}\,\theta + \frac{\theta^2}{2}\right) + \left(3\theta^2 + \cdots\right) + \cdots\right)$$

$$\approx \sqrt{3}\,a\left(1 + \sqrt{3}\,\theta + \frac{7}{2}\theta^2\right)$$

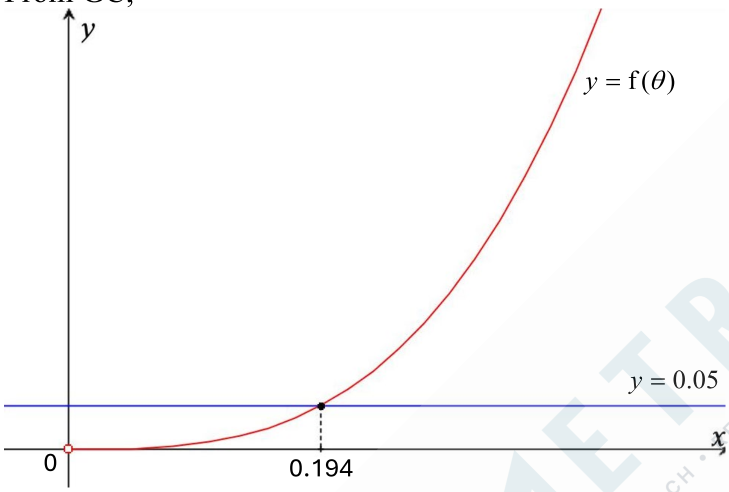
(iii)

$$\text{Let } f(\theta) = \left| \frac{\sqrt{3}\,a\left(1+\sqrt{3}\,\theta+\frac{7}{2}\theta^2\right) - \frac{\sqrt{3}\,a}{\cos\theta-\sqrt{3}\sin\theta}}{\frac{\sqrt{3}\,a}{\cos\theta-\sqrt{3}\sin\theta}} \right|$$

$$= \left| \frac{\left(1+\sqrt{3}\,\theta+\frac{7}{2}\theta^2\right) - \frac{1}{\cos\theta-\sqrt{3}\sin\theta}}{\frac{1}{\cos\theta-\sqrt{3}\sin\theta}} \right|$$

$\therefore f(\theta) < 0.05$

From GC,



$0 < \theta < 0.194 \quad (3\text{sf})$

12

(a)

$$\frac{AB}{\sin\frac{\pi}{6}}=\frac{1}{\sin\left(\frac{5\pi}{6}-x\right)}=\frac{1}{\sin\frac{5\pi}{6}\cos x-\cos\frac{5\pi}{6}\sin x}$$

$$\frac{AB}{\frac{1}{2}}=\frac{1}{\frac{1}{2}\cos x-\left(-\frac{\sqrt{3}}{2}\right)\sin x}$$

$$\frac{AB}{\frac{1}{2}}=\frac{1}{\frac{1}{2}\left(\cos x+\sqrt{3}\sin x\right)}$$

$$AB=\frac{1}{\cos x+\sqrt{3}\sin x} \quad \text{(Shown)}$$

(b)

$$AB\approx\frac{1}{\left(1-\frac{1}{2}x^{2}\right)+\sqrt{3}x}$$

$$\approx\left[1+\left(\sqrt{3}x-\frac{1}{2}x^{2}\right)\right]^{-1}$$

$$\approx 1-\left(\sqrt{3}x-\frac{1}{2}x^{2}\right)+\left(\sqrt{3}x-\frac{1}{2}x^{2}\right)^{2}$$

$$\approx 1-\sqrt{3}x+\frac{1}{2}x^{2}+3x^{2}$$

$$\approx 1-\sqrt{3}x+\frac{7}{2}x^{2}$$

(c)

$$\ln\left(px+q\right)=\ln\left[q\left(1+\frac{p}{q}x\right)\right]$$

$$=\ln q+\ln\left(1+\frac{p}{q}x\right)$$

$$=\ln q+\frac{p}{q}x+\cdots$$

Comparing, $\ln q=1$ and $\frac{p}{q}=-\sqrt{3}$

$q=\mathrm{e}$ and $p=-\sqrt{3}\mathrm{e}$

13

(i) $y=\cot(x+a)$

$$\frac{dy}{dx}=-\operatorname{cosec}^2(x+a)=-\left(1+\cot^2(x+a)\right)=-\left(1+y^2\right)$$

$$\frac{dy}{dx}=-1-y^2$$

$$\frac{d^2y}{dx^2}=-2y\frac{dy}{dx}=2y\left(1+y^2\right)=2y+2y^3$$

$$\frac{d^3y}{dx^3}=2\frac{dy}{dx}+6y^2\frac{dy}{dx}$$

$$=-2\left(1+y^2\right)-6y^2\left(1+y^2\right)$$

$$=-\left(6y^4+8y^2+2\right)$$

(ii) Since $\tan 0=0$, $\cot 0=\frac{1}{\tan 0}$ is undefined.

Hence the Maclaurin series of $\cot(x+0)$ cannot be found.

(iii) $y=\cot\left(x+\frac{\pi}{2}\right)$

When $x=0$, $y=\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}=\frac{0}{1}=0$, $\frac{dy}{dx}=-1-0^2=-1$,

$$\frac{d^2y}{dx^2}=2(0)+2(0)^3=0,\ \frac{d^3y}{dx^3}=-\left(6(0)^4+8(0)^2+2\right)=-2$$

$$y\approx -x-\frac{2}{3!}x^3=-x-\frac{x^3}{3}$$

(iv)

$$\frac{y}{2+x}=y(2+x)^{-1}$$

$$=y\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$$

$$=\frac{1}{2}y\left(1+\frac{x}{2}\right)^{-1}$$

$$\approx\frac{1}{2}\left(-x-\frac{1}{3}x^3\right)\left(1+(-1)\frac{x}{2}+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2\right)$$

$$=\frac{1}{2}\left(-x-\frac{1}{3}x^3\right)\left(1-\frac{1}{2}x+\frac{1}{4}x^2\right)$$

$$\approx\frac{1}{2}\left(-x-\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{1}{4}x^3\right)$$

$$=-\frac{1}{2}x+\frac{1}{4}x^2-\frac{7}{24}x^3$$

14

(a) $y = (1+\tan^{-1} x)^2$

Differentiating with respect to x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1+\tan^{-1} x)\left(\frac{1}{1+x^2}\right)$$

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1+\tan^{-1} x)$$

Differentiating with respect to x again,

$$(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{1}{1+x^2}\right)$$

$$(1+x^2)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \text{(shown)}$$

(b) Differentiate $(1+x^2)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ with respect to x,

$$(1+x^2)^2\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 4x(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (2+6x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2x(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$$

$$(1+x^2)^2\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6x(1+x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (2+6x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

(c) When $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2$, $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -4$

Hence the Maclaurin's series for y is

$$y \approx 1 + 2x + x^2 - \frac{2}{3}x^3.$$

(d) Using the Maclaurin's Series for y, $y \approx 1 + 2x + x^2 - \frac{2}{3}x^3$.

It was observed that the gradient of the tangent to the curve at $x = 0$ is 2. Hence the gradient of normal is $-\frac{1}{2}$. From (c), $y = 1$ when $x = 0$.

Therefore, the equation of normal to the curve at $x = 0$ is $y = -\frac{1}{2}x + 1$.

15

(i) $\left(1-x^2\right)f''(x)-xf'(x)=0$ (Given)

Differentiating,

$\left(1-x^2\right)f'''(x)-2xf''(x)-xf''(x)-f'(x)=0$

At $x=0$, substitute into:

$\left(1-x^2\right)f''(x)-xf'(x)=0$

$f''(0)-0=0$

$f''(0)=0$

At $x=0$, substitute into:

$\left(1-x^2\right)f'''(x)-2xf''(x)-xf''(x)-f'(x)=0$

$f'''(0)-0-0-f'(0)=0$

$f'''(0)=-1$

Thus,

$f(x)$

$=f(0)+xf'(0)+\dfrac{x^2}{2!}f''(0)+\dfrac{x^3}{3!}f'''(0)+\ldots$

$=1+x(-1)+\dfrac{x^2}{2!}(0)+\dfrac{x^3}{3!}(-1)+\ldots$

$=1-x-\dfrac{x^3}{6}+\ldots$

(ii) $f(x)=1-x-\dfrac{x^3}{6}+\ldots$

$f'(x)=-1-\dfrac{x^2}{2}+\ldots$

$\dfrac{f'(x)}{f(x)}=\dfrac{-1-\dfrac{x^2}{2}+\ldots}{1-x-\dfrac{x^3}{6}+\ldots}$

$=\left(-1-\dfrac{x^2}{2}+\ldots\right)\left(1-x-\dfrac{x^3}{6}+\ldots\right)^{-1}$

$=\left(-1-\dfrac{x^2}{2}+\ldots\right)(1-x+\ldots)^{-1}$ (Need not consider x^3-term anymore.)

$$= \left(-1 - \frac{x^2}{2} + \ldots\right)\left(1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 \ldots\right)$$

$$= \left(-1 - \frac{x^2}{2} + \ldots\right)\left(1 + x + x^2 \ldots\right)$$

$$= \left(-1 - x - x^2 - \frac{x^2}{2} + \ldots\right)$$

$$= -1 - x - \frac{3}{2}x^2 + \ldots$$

(iii) $\frac{f'(x)}{f(x)} = -1 - x - \frac{3}{2}x^2 + \ldots$

Integrating both sides with respect to *x*,

$$\int \frac{f'(x)}{f(x)}\, dx = \int -1 - x - \frac{3}{2}x^2 + \ldots\ldots\, dx$$

$$\ln|f(x)| = \left[-x - \frac{1}{2}x^2 + \ldots\right] + C$$

When $x = 0$, $f(0) = 1$:

$$\ln|1| = C \Rightarrow C = 0$$

$$\ln|f(x)| = -x - \frac{1}{2}x^2 + \ldots$$

16

(a)

$$y=\frac{1}{3+\sin 2x}$$

$$y(3+\sin 2x)=1$$

Differentiate implicitly w.r.t. x:

$$(3+\sin 2x)\left(\frac{dy}{dx}\right)+y(2\cos 2x)=0$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right)+2y\cos 2x=0$$

$$\frac{dy}{dx}+2y^2\cos 2x=0$$

Alternatively

$$y=(3+\sin 2x)^{-1}$$

$$\frac{dy}{dx}=-(3+\sin 2x)^{-2}(2\cos 2x)$$

$$\frac{dy}{dx}=-y^2(2\cos 2x)$$

Differentiate implicitly w.r.t. x:

$$\frac{d^2y}{dx^2}-2y^2 2\sin 2x+2(2y)\frac{dy}{dx}\cos 2x=0$$

$$\frac{d^2y}{dx^2}-4y^2\sin 2x+4y\frac{dy}{dx}\cos 2x=0$$

Given that $x=0$, $y=\frac{1}{3+0}=\frac{1}{3}$,

$$\frac{dy}{dx}=-2\left(\frac{1}{3}\right)^2\cos 0=-\frac{2}{9},$$

$$\frac{d^2y}{dx^2}=4\left(\frac{1}{3}\right)^2\sin 0-4\left(\frac{1}{3}\right)\left(-\frac{2}{9}\right)\cos 0=\frac{8}{27}$$

$$y=\frac{1}{3}-\frac{2}{9}x+\frac{x^2}{2!}\left(\frac{8}{27}\right)+\ldots\approx\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2$$

(b)

$$y = \frac{1}{3+\sin 2x}$$

$$\approx \frac{1}{3+\left(2x+\cdots\right)}$$

$$= \left(3+2x\right)^{-1}$$

$$= \left[3\left(1+\frac{2}{3}x\right)\right]^{-1}$$

$$= \frac{1}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^2 \ldots\right)$$

$$= \frac{1}{3}\left(1-\frac{2}{3}x+\frac{4}{9}x^2\ldots\right)$$

$$\approx \frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2 \text{ (verified)}$$

(c)

$$\int_0^{1.5} \frac{1}{3+\sin 2x}\,\mathrm{d}x$$

$$\approx \int_0^{1.5}\left(\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2\right)\mathrm{d}x$$

$$= \left[\frac{1}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3\right]_0^{1.5}$$

$$= \frac{5}{12} \text{ or } 0.417 \text{ (3.s.f)}$$

17

(a)

$$\sin 3x = (3x) - \frac{(3x)^3}{3!} + \ldots = 3x - \frac{9x^3}{2} + \ldots$$

$$\mathrm{f}(x) = \mathrm{e}^{\sin 3x} = 1 + (\sin 3x) + \frac{(\sin 3x)^2}{2!} + \frac{(\sin 3x)^3}{3!} + \ldots$$

$$= 1 + (3x - \frac{9x^3}{2} + \ldots) + \frac{1}{2}(3x - \frac{9x^3}{2} + \ldots)^2 + \frac{1}{6}(3x - \frac{9x^3}{2} + \ldots)^3 + \ldots$$

$$= 1 + 3x - \frac{9x^3}{2} + \frac{1}{2}[(3x)^2] + \frac{1}{6}[(3x)^3] + \ldots$$

$$\approx 1 + 3x + \frac{9}{2}x^2 + 0x^3 \quad \text{(independent of } x^3\text{)}$$

Alternative (by differentiation)

let $y = \mathrm{e}^{\sin 3x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos 3x \cdot \mathrm{e}^{\sin 3x} = 3\cos 3x \cdot y$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3\cos 3x \frac{\mathrm{d}y}{\mathrm{d}x} - 9\sin 3x \cdot y$$

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 3\cos 3x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 9\sin 3x \frac{\mathrm{d}y}{\mathrm{d}x} - 9\sin 3x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - 27\cos 3x \cdot y$$

When $x = 0$,

$$y = 1,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 3,\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 9,\ \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 0$$

$$\therefore y = 1 + 3x + \frac{9}{2}x^2 + 0x^3 + \ldots$$

(b)

$$\int \frac{\mathrm{e}^{\sin 3x}}{x^2}\,\mathrm{d}x \approx \int (x^{-2} + 3x^{-1} + \frac{9}{2})\,\mathrm{d}x$$

$$= -x^{-1} + 3\ln|x| + \frac{9}{2}x + C \quad \text{where } C \text{ is an arbitrary constant}$$

$$\int_{0.1}^{0.2} (\frac{2}{x})^2 \mathrm{e}^{\sin 3x}\,\mathrm{d}x = \int_{0.1}^{0.2} \frac{4\mathrm{e}^{\sin 3x}}{x^2}\,\mathrm{d}x$$

$$= 4[-x^{-1} + 3\ln x + \frac{9}{2}x]_{0.1}^{0.2}$$

$$= 30.1178 \text{ (4 d.p.)}$$

(c) Using GC, $\int_{0.1}^{0.2} (\frac{2}{x})^2 \mathrm{e}^{\sin 3x}\,\mathrm{d}x = 29.9995$ (4 d.p.)

(d) The **approximation is accurate** as the **values of *x*** (between 0.1 and 0.2) **are close to 0** for the magnitude of x^4 and higher powers of x to be neglected.

Alternative:

$$\%\text{ error } = \frac{|30.1178 - 29.9995|}{29.9995} \times 100 = 0.3943\%$$

Since percentage error is small, approximation is accurate.

18

(a)
$$\frac{1}{\sqrt{1-a^2x^2}}=\left(1+\left(-a^2x^2\right)\right)^{-\frac{1}{2}}$$
$$=1+\left(-\frac{1}{2}\right)\left(-a^2x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-a^2x^2\right)^2+\ldots$$
$$=1+\frac{a^2}{2}x^2+\frac{3a^4}{8}x^4+\ldots$$

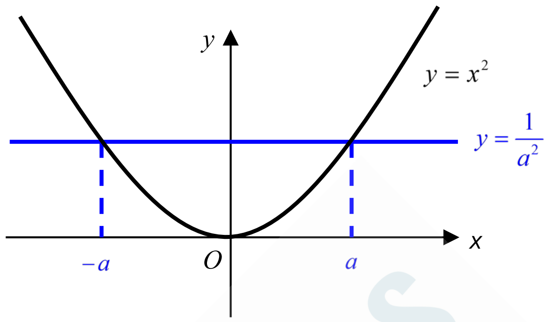
The expansion is valid for

$\left|-a^2x^2\right|<1$

$a^2x^2<1$ since $a^2x^2\geq 0$

$x^2<\frac{1}{a^2}$ since a is positive

From graph, $-\frac{1}{a}<x<\frac{1}{a}$



(b) Since $\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\cos^{-1}x+C$, substituting $a=1$ into the expansion obtained in **(a)** and integrating,

$\cos^{-1}x\approx-\int 1+\frac{x^2}{2}+\frac{3}{8}x^4\,\mathrm{d}x$

$\cos^{-1}x\approx c-x-\frac{x^3}{6}-\frac{3}{40}x^5$ for some constant c

Since $\cos^{-1}0=\frac{\pi}{2}$, substituting $x=0$, $c=\frac{\pi}{2}$.

$\therefore \cos^{-1}x=\frac{\pi}{2}-\frac{x^3}{6}-\frac{3}{40}x^5+\ldots$

(c) $\int_0^{\frac{1}{2}}\cos^{-1}x\,\mathrm{d}x\approx\int_0^{\frac{1}{2}}\frac{\pi}{2}-x-\frac{x^3}{6}-\frac{3x^5}{40}\,\mathrm{d}x=0.65760\ (5sf)$

(d) By GC, $\int_0^{\frac{1}{2}}\cos^{-1}x\,\mathrm{d}x=0.65757\ (5sf)$

Estimate in **(c)** is accurate to 4sf but not to 5sf.

To improve estimate, we can include higher-order terms in the Maclaurin series expansion of $\cos^{-1}x$