1

(a) $x^3 + y^3 - xy = A$

Differentiating with respect to x,

$3x^2 + 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$

$\left(3y^2 - x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2 + y$ (*)

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$,

$0 = -3x^2 + y$

$y = 3x^2$ (shown)

(b) At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = 3x^2$.

Sub $y = 3x^2$ into $x^3 + y^3 - xy = A$,

$x^3 + \left(3x^2\right)^3 - x\left(3x^2\right) = A$

$x^3 + 27x^6 - 3x^3 = A$

$27x^6 - 2x^3 - A = 0$

$x^3 = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(27)(-A)}}{2(27)}$

$x^3 = \frac{2 \pm 2\sqrt{1+27A}}{2(27)}$

$x^3 = \frac{1 \pm \sqrt{1+27A}}{27}$

$x = \sqrt[3]{\frac{1 \pm \sqrt{1+27A}}{27}} = \frac{1}{3}\sqrt[3]{1 \pm \sqrt{1+27A}}$

For more than one stationary point,
Discriminant $1 + 27A > 0$

So $A > -\frac{1}{27}$

(c) From (*), $\left(3y^2 - x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2 + y$

Differentiate with respect to x,

$$\left(3y^2 - x\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(6y\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -6x + \frac{\mathrm{d}y}{\mathrm{d}x}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = 3x^2$, so we have

$$\left(3\left(3x^2\right)^2 - x\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -6x$$

$$\left(27x^4 - x\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -6x$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{6x}{27x^4 - x} = -\frac{6}{27x^3 - 1}$$

When $x^3 = \frac{1 + \sqrt{1 + 27A}}{27}$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{6}{27x^3 - 1} = -\frac{6}{\sqrt{1 + 27A}} < 0$$ so this is a maximum point.

When $x^3 = \frac{1 - \sqrt{1 + 27A}}{27}$,

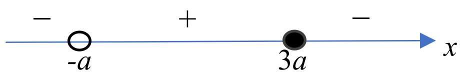
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{6}{27x^3 - 1} = \frac{6}{\sqrt{1 + 27A}} > 0$$ so this is a minimum point.

2

(a) $\dfrac{y^2}{x^2} = \dfrac{3a - x}{a + x}$, $x \neq 0$

Since $\dfrac{y^2}{x^2} \geq 0$ for all values of x and y,

$\dfrac{3a - x}{a + x} \geq 0$, $x \neq -a$



$-a < x \leq 3a$

Since $x \neq 0$,

$\therefore -a < x < 0$ or $0 < x \leq 3a$

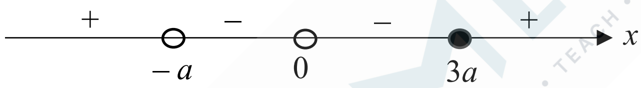
<u>Alternative Solution</u>

$y^2 = \dfrac{(3a - x)x^2}{a + x}$, $x \neq 0$

Since $y^2 \geq 0$, $\dfrac{(3a - x)x^2}{a + x} \geq 0$, $x \neq 0$

$-\dfrac{(x - 3a)x^2}{a + x} \geq 0$

$\dfrac{(x - 3a)x^2}{x + a} \leq 0$



$\therefore -a < x < 0$ or $0 < x \leq 3a$

(b) Let $a = 1$, we have $y^2(1 + x) = x^2(3 - x)$.

$$2y\frac{dy}{dx}(1 + x) + y^2 = 2x(3 - x) - x^2$$

$\dfrac{dy}{dx} = 0$ when tangent is parallel to the x-axis.

When $\dfrac{dy}{dx} = 0$, $y^2 = 2x(3 - x) - x^2$

$$= 6x - 3x^2$$

Substitute $y^2 = 6x - 3x^2$ in $y^2(1 + x) = x^2(3 - x)$,

$$(6x - 3x^2)(1 + x) = x^2(3 - x)$$

$$6x + 6x^2 - 3x^2 - 3x^3 = 3x^2 - x^3$$

$$x^3 - 3x = 0$$

$$x(x^2 - 3) = 0$$

$$x = 0 \text{ or } x = \pm\sqrt{3}$$

Since $x \neq 0$ and $-a < x \leq 3a$, we have $x = \sqrt{3}$

When $x = \sqrt{3}$, $y^2 = 6\sqrt{3} - 3(3)$

$$y = \pm\sqrt{6\sqrt{3} - 9}$$

The coordinates of the points are $\left(\sqrt{3}, \sqrt{6\sqrt{3} - 9}\right)$ and $\left(\sqrt{3}, -\sqrt{6\sqrt{3} - 9}\right)$

(c) When $x = 1$,

$$y^2(1+1) = (1)^2(3-1)$$
$$y = \pm 1$$

When $x = 1$, $y = 1$,

$$2\frac{\mathrm{d}y}{\mathrm{d}x}(1+1) + 1^2 = 2(3-1) - 1^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$$

Equation of normal is $y - 1 = -2(x-1)$

$$\Rightarrow y = -2x + 3$$

When $x = 1$, $y = -1$,

$$-2\frac{\mathrm{d}y}{\mathrm{d}x}(1+1) + 1^2 = 2(3-1) - 1^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$$

Equation of normal is $y - (-1) = 2(x-1)$

$$\Rightarrow y = 2x - 3$$

Solving $y = -2x + 3$ and $y = 2x - 3$, we get $x = \frac{3}{2}$ and $y = 0$

Therefore, the coordinates of N are $\left(\frac{3}{2}, 0\right)$

3

$y=1-\frac{1}{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=x^{-2}$

At $x=a$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{a^2}, y=1-\frac{1}{a}$

Tangent at $x=a$

$y-\left(1-\frac{1}{a}\right)=\frac{1}{a^2}(x-a)$

$y-1+\frac{1}{a}=\frac{1}{a^2}(x-a)$

$a^2y-a^2+a=x-a$

$a^2y-x=a^2-2a\,(\text{Shown})$

$9y-x=1 \;\Rightarrow\; y=\frac{1}{9}(x+1)$

If l is parallel to $9y-x=1$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{a^2}=\frac{1}{9}$

$a=-3$ (reject as $a>0$) or 3

When $a=3$,

$3^2y-x=3^2-2(3)$

$9y-x=3$

4

(a)(i) $\dfrac{dy}{dx}=\dfrac{(3-\sin 2x)(-8\sin 2x)-4\cos 2x(-2\cos 2x)}{(3-\sin 2x)^2}$

$=\dfrac{-24\sin 2x+8\sin^2 2x+8\cos^2 2x}{(3-\sin 2x)^2}$

$=\dfrac{-24\sin 2x+8}{(3-\sin 2x)^2}$, since $\sin^2 2x+\cos^2 2x=1$

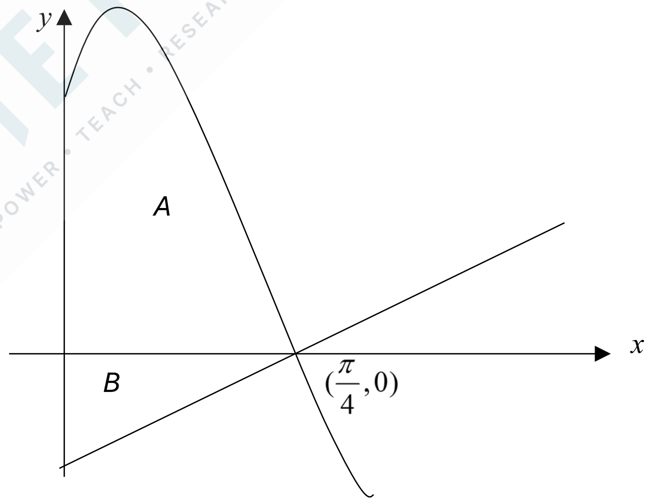
At $\left(\dfrac{\pi}{4},0\right)$, $\dfrac{dy}{dx}=\dfrac{-24\sin 2\left(\dfrac{\pi}{4}\right)+8}{\left(3-\sin 2\left(\dfrac{\pi}{4}\right)\right)^2}=-4$

Equation of normal:

$y-0=\dfrac{1}{4}(x-\dfrac{\pi}{4})$

$y=\dfrac{1}{4}x-\dfrac{\pi}{16}$

(ii)



$u=\sin 2x$

$\dfrac{du}{dx}=2\cos 2x$

$du=2\cos 2x\ dx$

When $x=0$, $u=0$;

When $x=\dfrac{\pi}{4}$, $u=\sin\left(\dfrac{\pi}{2}\right)=1$

<u>Method 1</u>

Area A

$=\int_0^{\frac{\pi}{4}}\dfrac{4\cos 2x}{3-\sin 2x}\,dx$

$=\int_0^{\frac{\pi}{4}}\dfrac{2}{3-\sin 2x}2\cos 2x\ dx$

$=\int_0^{1}\dfrac{2}{3-u}\,du$

$=[-2\ln|3-u|]_0^1=-2\ln 2+2\ln 3=2\ln\dfrac{3}{2}$

Area B

$=\dfrac{1}{2}\left(\dfrac{\pi}{4}\right)\left(\dfrac{\pi}{16}\right)=\dfrac{\pi^2}{128}$

Thus total area required = Area A + Area B = $2\ln\dfrac{3}{2}+\dfrac{\pi^2}{128}$

Method 2

$u = \sin 2x$

$\frac{du}{dx} = 2\cos 2x$

$du = 2\cos 2x\ dx$

When $x = 0$, $u = 0$;

When $x = \frac{\pi}{4}, u = \sin\left(\frac{\pi}{2}\right) = 1$

Area

$= \int_0^{\frac{\pi}{4}} \frac{4\cos 2x}{3 - \sin 2x} dx - \int_0^{\frac{\pi}{4}} \frac{1}{4}x - \frac{\pi}{16} dx$

$= \int_0^{\frac{\pi}{4}} \frac{2}{3 - \sin 2x} 2\cos 2x\ dx - \int_0^{\frac{\pi}{4}} \frac{1}{4}x - \frac{\pi}{16} dx$

$= \int_0^{1} \frac{2}{3 - u} du - \int_0^{\frac{\pi}{4}} \frac{1}{4}x - \frac{\pi}{16} dx$

$= [-2\ln|3-u|]_0^1 - \left[\frac{x^2}{8} - \frac{\pi x}{16}\right]_0^{\frac{\pi}{4}}$

$= -2\ln 2 + 2\ln 3 + \frac{\pi^2}{128}$

$= 2\ln\frac{3}{2} + \frac{\pi^2}{128}$

(b) $\frac{dy}{dx} = \frac{-24\sin 2x + 8}{(3 - \sin 2x)^2}$

When $\frac{\pi}{12} < x < \frac{5\pi}{12}$,

$\frac{\pi}{6} < 2x < \frac{5\pi}{6}$

$\frac{1}{2} < \sin 2x \le 1$

$-24 \le -24\sin 2x < -12$

$-16 \le -24\sin 2x + 8 < -4$

At the same time, $(3 - \sin 2x)^2 > 0$, thus $\frac{dy}{dx} = \frac{-24\sin 2x + 8}{(3 - \sin 2x)^2} < 0$ when $\frac{\pi}{12} < x < \frac{5\pi}{12}$

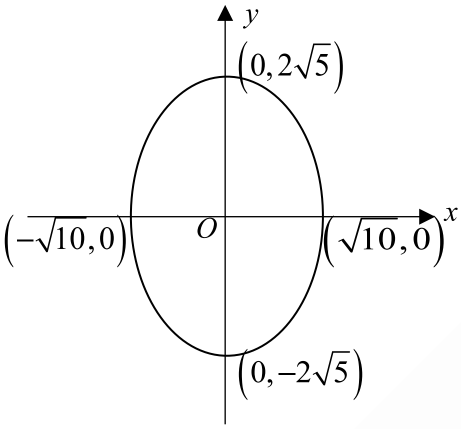
5

(i) $2x^2 + y^2 = 20$

$\frac{x^2}{10} + \frac{y^2}{20} = 1$

$\frac{x^2}{\left(\sqrt{10}\right)^2} + \frac{y^2}{\left(\sqrt{20}\right)^2} = 1$

$\frac{x^2}{\left(\sqrt{10}\right)^2} + \frac{y^2}{\left(2\sqrt{5}\right)^2} = 1$



(ii) $2x^2 + y^2 = 20$

$4x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -4x$

$\frac{dy}{dx} = -\frac{4x}{2y} = -\frac{2x}{y}$ (shown)

(iii) Method 1

Gradient of normal $= \frac{y}{2x}$

At (a, b), Gradient of normal $= \frac{b}{2a}$

$\frac{b}{2a} = \frac{b-0}{a-1}$

$ab - b = 2ab$

$ab + b = 0$

$b(a+1) = 0$

$b = 0$ or $a = -1$

When $b = 0$, $2a^2 + (0)^2 = 20$

$a = -\sqrt{10}$ or $\sqrt{10}$

or

When $a=-1$, $2(-1)^2+b^2=20$

$b^2=18$

$b=-\sqrt{18}$ or $\sqrt{18}$

$b=-3\sqrt{2}$ or $3\sqrt{2}$

The four coordinates are $\left(\sqrt{10},0\right),\left(-\sqrt{10},0\right),\left(-1,-3\sqrt{2}\right)$ and $\left(-1,3\sqrt{2}\right)$.

Method 2

Gradient of normal $=\dfrac{y}{2x}$

At (a,b), Gradient of normal $=\dfrac{b}{2a}$

Equation of normal at $P: y-b=\dfrac{b}{2a}(x-a)$

$$y=\frac{b}{2a}x-\frac{b}{2}+b$$

$$y=\frac{b}{2a}x+\frac{b}{2}$$

Since normal passes through (1, 0),

$0=\dfrac{b}{2a}(1)+\dfrac{b}{2}$

$0=b+ab$

$b(1+a)=0$

$b=0$ or $a=-1$

When $b=0$, $2a^2+(0)^2=20$

$a=-\sqrt{10}$ or $\sqrt{10}$

or

When $a=-1$, $2(-1)^2+b^2=20$

$b^2=18$

$b=-\sqrt{18}$ or $\sqrt{18}$

$b=-3\sqrt{2}$ or $3\sqrt{2}$

The four coordinates are $\left(\sqrt{10},0\right),\left(-\sqrt{10},0\right),\left(-1,-3\sqrt{2}\right)$ and $\left(-1,3\sqrt{2}\right)$.

6

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiate w.r.t x:

$$\frac{2x}{a^2}+\frac{2y}{b^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$$

$$\frac{2y}{b^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\frac{2x}{a^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{b^2x}{a^2y}\ (\text{since } y\neq 0)(\text{shown})$$

(b) At $P(a\cos\theta, b\sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{b^2(a\cos\theta)}{a^2(b\sin\theta)}=-\frac{b\cos\theta}{a\sin\theta}$$

Equation of tangent at P,

$$y-(b\sin\theta)=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$\frac{y}{b}-\sin\theta=-\frac{\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$\frac{y}{b}\sin\theta-\sin^2\theta=-\frac{\cos\theta}{a}(x-a\cos\theta)$$

$$\frac{y}{b}\sin\theta-\sin^2\theta=-\frac{x}{a}\cos\theta+\cos^2\theta$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\sin^2\theta+\cos^2\theta$$

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1\ (\text{shown})$$

(c) When $x=0$,

$$\frac{y}{b}\sin\theta=1\Rightarrow y=\frac{b}{\sin\theta}$$

When $y=0$,

$$\frac{x}{a}\cos\theta=1\Rightarrow x=\frac{a}{\cos\theta}$$

Area of triangle ORS

$$=\frac{1}{2}\left(\frac{b}{\sin\theta}\right)\left(\frac{a}{\cos\theta}\right)=\frac{ab}{2\sin\theta\cos\theta}=\frac{ab}{\sin 2\theta}$$

(d)

$0 < \theta < \frac{\pi}{2}$

$0 < \sin 2\theta \le 1$

$\frac{1}{\sin 2\theta} \ge 1$

$\frac{ab}{\sin 2\theta} \ge ab$, since a and b are positive

This means minimum area is ab when $\sin 2\theta = 1$.

$\sin 2\theta = 1$

$\theta = \frac{\pi}{4} \quad \left(\text{since } 0 < \theta < \frac{\pi}{2}\right)$

Therefore, minimum area is ab (shown) and occurs when $\theta = \frac{\pi}{4}$.

Alternative method

Let area of triangle ORS be A.

$$A = \frac{ab}{\sin 2\theta} = ab(\sin 2\theta)^{-1}$$

$$\frac{dA}{d\theta} = -ab(\sin 2\theta)^{-2}(2\cos 2\theta) = \frac{-2ab\cos 2\theta}{(\sin 2\theta)^2}$$

At stationary point, $\frac{dA}{d\theta} = 0$

$$\frac{-2ab\cos 2\theta}{(\sin 2\theta)^2} = 0$$

$$\cos 2\theta = 0$$

$$\theta = \frac{\pi}{4} \quad (\because \ \theta \text{ is acute})$$

Minimum area of triangle ORS

$$A = \frac{ab}{\sin 2\left(\frac{\pi}{4}\right)} = ab$$

Therefore, minimum area is ab and occurs when $\theta = \frac{\pi}{4}$.

7

(a) <u>**Method 1:**</u>

$x^2+y^2-8x-6y-20=0$ crosses positive x axis at A:

Let $y=0$:

$x^2-8x-20=0$

$(x-10)(x+2)=0$

$x=10$ or $x=-2$ (Rejected as $x>0$)

$\therefore A(10,0)$

$x^2+y^2-8x-6y-20=0$

$$2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}-8-6\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{8-2x}{2y-6}=\frac{4-x}{y-3}$$

Tangent at A:

$$y-0=\frac{4-10}{0-3}(x-10)$$

$y=2x-20$

<u>**Method 2:**</u>

$x^2+y^2-8x-6y-20=0$

$(x-4)^2-4^2+(y-3)^2-3^2-20=0$

$(x-4)^2+(y-3)^2=45$

When $y=0$,

$(x-4)^2=36$

$x=-2$ (Rejected $\because x>0$) or $x=10$

$\therefore A(10,0)$

$$2(x-4)+2(y-3)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$2(10-4)+2(0-3)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2$$

Tangent at A:

$y-0=2(x-10)$

$y=2x-20$

(b) $x^2+y^2-8x-6y-20=0$ crosses positive y axis at B:

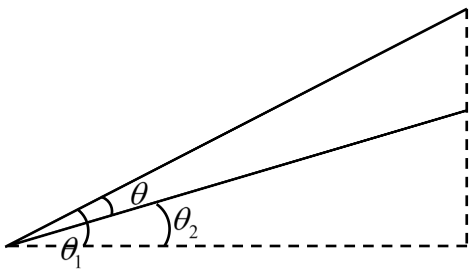
Set $x=0$,

$y^2-6y-20=0$

$y=3+\sqrt{29}=8.3852$ or $y=3-\sqrt{29}=-2.3852$ $\left(\text{rejected } \because y>0\right)$

$\therefore B\left(0,8.3852\right)$

$$\left.\frac{dy}{dx}\right|_{x=0}=\frac{4-x}{y-3}=\frac{4}{8.3852-3}=0.74278=0.743$$



Acute angle between both tangents $=\theta_1-\theta_2$

$=\tan^{-1}2-\tan^{-1}0.74278=26.8^o$

(c) $x^2+y^2-8x-6y-20=0$

$(x-4)^2-16+y^2-6y-20=0$

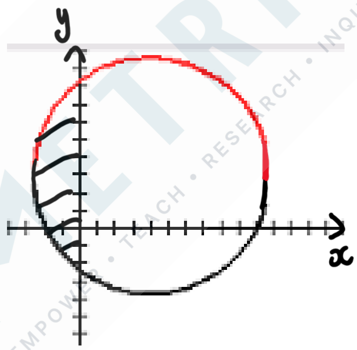
$(x-4)^2=-y^2+6y+36$

$x=4\pm\sqrt{-y^2+6y+36}$

since $x<0$, $x=4-\sqrt{-y^2+6y+36}$

Volume generated

$=\pi\int_{-2.3852}^{8.3852}(4-\sqrt{-y^2+6y+36})^2\,dy=141.5$



8

(i) $2xy + x - 9y = 0$

Differentiate with respect x,

$$2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y + 1 - 9\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(2x-9) = -2y-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y+1}{9-2x}$$

(ii) Let $G = \frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\therefore G = \frac{2y+1}{9-2x}$$

Diff wrt x,

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \frac{2\frac{\mathrm{d}y}{\mathrm{d}x}(9-2x) - (2y+1)(-2)}{(9-2x)^2}$$

When $x = 3$,

$$2xy + x - 9y = 0$$

$$\Rightarrow 6y + 3 - 9y = 0$$

$$\Rightarrow y = 1$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(1)+1}{9-2(3)} = 1$$

Hence, when $x = 3$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= \frac{2(1)(9-2(3)) - (2(1)+1)(-2)}{(9-2(3))^2} \times 0.02$$

$$= \frac{2}{75} \quad \text{or} \quad 0.0267 \text{ (3sf)}$$

Therefore, required rate is $\frac{2}{75}$ units/s. (or 0.0267 units/s)

9

(a) We have $A = 4\pi r^2 \Rightarrow \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$.

Hence $-100 = 8\pi(5)\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -0.796$ cm/s

Alternative

$A = 4\pi r^2 \Rightarrow \frac{dA}{dr} = 8\pi r$

$$\frac{dr}{dt} = \frac{dA}{dt} \times \frac{dr}{dA}$$
$$= -100 \times \frac{1}{8\pi(5)}$$
$$= -\frac{5}{2\pi}$$

Hence the radius is decreasing at $\frac{5}{2\pi}$ cm/s.

(b) Volume of meteorite, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Since V decreases with t, we have $\frac{dV}{dt} = -kA$ for proportionality constant $k > 0$.

This means that $4\pi r^2 \frac{dr}{dt} = -k\left(4\pi r^2\right) \Rightarrow \frac{dr}{dt} = -k$, which is a negative constant.

Hence the radius is decreasing at a constant rate.

Alternative

We have $\frac{dV}{dt} = -kA = -k(4\pi r^2)$ for proportionality constant $k > 0$.

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$$
$$= -k(4\pi r^2) \times \frac{1}{4\pi r^2}$$
$$= -k < 0$$

Since k is a negative constant, thus the radius is decreasing at a constant rate.

10

(a)

$$A=\frac{1}{2}(PQ)(PR)\sin\angle QPR$$

$$=\frac{1}{2}\left(x+1\right)\left(4-x\right)^{2}\sin 30°$$

$$=\frac{1}{4}(x+1)(4-x)^{2}$$

$$=\frac{1}{4}(x+1)(16-8x+x^{2})$$

$$=\frac{1}{4}(16x-8x^{2}+x^{3}+16-8x+x^{2})$$

$$=\frac{1}{4}(x^{3}-7x^{2}+8x+16) \text{ (Shown)}$$

Or let N be the foot of perpendicular from Q to PR.

$$QN=PQ\sin 30^{\circ}=\frac{1}{2}\left(x+1\right)$$

$$A=\frac{1}{2}(QN)(PR)$$

$$=\frac{1}{4}(x+1)(4-x)^{2}$$

$$=\frac{1}{4}(x+1)(16-8x+x^{2})$$

$$=\frac{1}{4}(16x-8x^{2}+x^{3}+16-8x+x^{2})$$

$$=\frac{1}{4}(x^{3}-7x^{2}+8x+16) \text{ (Shown)}$$

(b) $\dfrac{dA}{dx} = \dfrac{1}{4}(3x^2 - 14x + 8)$

At stationary values, $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad \dfrac{1}{4}(3x^2 - 14x + 8) = 0$

$\Rightarrow \quad x = \dfrac{2}{3}$ or $x = 4$ (rejected since it is given that $x < 4$)

$\left.\dfrac{d^2A}{dx^2}\right|_{x=\frac{2}{3}} = \left.\dfrac{1}{4}(6x - 14)\right|_{x=\frac{2}{3}} = -2.5 < 0$ (maximum)

To find QR:

When $x = \dfrac{2}{3}$, $PQ = \dfrac{5}{3}$ and $PR = \dfrac{100}{9}$

Using cosine rule,

$$QR^2 = \left(\frac{5}{3}\right)^2 + \left(\frac{100}{9}\right)^2 - 2\left(\frac{5}{3}\right)\left(\frac{100}{9}\right)\cos 30^\circ$$

$\therefore \quad QR \approx 9.70$ (3 s.f.)

Or $QN = \dfrac{1}{2}\left(\dfrac{2}{3} + 1\right) = \dfrac{5}{6}$

$PN = PQ\cos 30^\circ = \dfrac{\sqrt{3}}{2}\left(\dfrac{5}{3}\right) = \dfrac{5\sqrt{3}}{6}$

$RN = PR - PN = \dfrac{100}{9} - \dfrac{5\sqrt{3}}{6}$

$$QR = \sqrt{QN^2 + RN^2} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{100}{9} - \frac{5\sqrt{3}}{6}\right)^2} = 9.70$$

11

(a)

$$y^2 = h^2 + \left(\frac{x}{2}\right)^2$$
$$= h^2 + \left(\frac{20-2y}{2}\right)^2$$
$$= h^2 + \left(100 - 20y + y^2\right)$$
$$20y = 100 + h^2$$
$$y = 5 + \frac{h^2}{20} \quad \text{(shown)}$$

$$x + 2y = 20$$
$$x = 20 - 2y$$

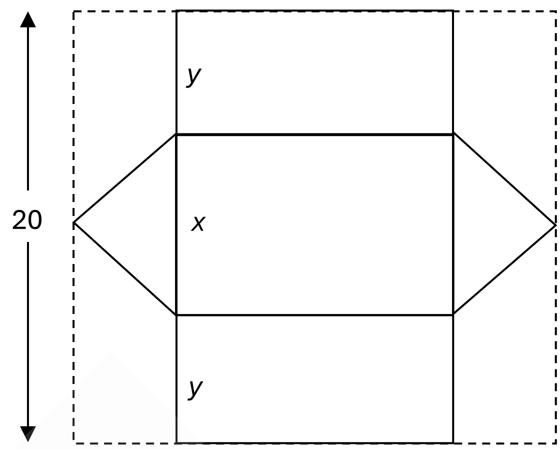


Fig. 1

$$x = 20 - 2y$$
$$= 20 - 2\left(5 + \frac{h^2}{20}\right)$$
$$= 10 - \frac{h^2}{10}$$

(b) Volume of prism,

$$V = \frac{1}{2}hxz$$
$$= \frac{1}{2}h\left(10 - \frac{h^2}{10}\right)(20 - 2h)$$
$$= \frac{h}{10}\left(100 - h^2\right)(10 - h)$$
$$= \frac{h}{10}\left(1000 - 100h - 10h^2 + h^3\right)$$
$$= \frac{1}{10}\left(h^4 - 10h^3 - 100h^2 + 1000h\right) \text{ (shown)}$$

$$\frac{dV}{dh} = \frac{1}{10}\left(4h^3 - 30h^2 - 200h + 1000\right)$$

For max.volume,

$$\frac{dV}{dh} = 0 \Rightarrow 4h^3 - 30h^2 - 200h + 1000 = 0$$

From GC : $h = -6.40$ or 10 (reject $\because 0 < h < 10$)

$$\therefore h = 3.9039$$

$$\frac{d^2V}{dh^2} = \frac{1}{10}(12h^2 - 60h - 200)$$

When $h = 3.9039$,

$$\frac{d^2V}{dh^2} = \frac{1}{10}(12(3.9039)^2 - 60(3.9039) - 200) = -25.1 < 0$$

$\therefore h = 3.9039$ gives maximum volume

Max volume $= \frac{1}{10}\left(3.9039^4 - 10(3.9039)^3 - 100(3.9039)^2 + 1000(3.9039)\right)$

$= 201.71 = 202 \text{cm}^3$ (3 sf)

12

(a)

$\sin\theta = \dfrac{d}{PQ}$

$PQ = d\ \text{cosec}\,\theta$

$PQ + QR + RS = 2a$

$QR = 2a - 2(d\text{cosec}\,\theta)$

$QR = 2(a - d\text{cosec}\,\theta)$

$$\text{Area of trapezium } PQRS = \frac{1}{2}d(QR + PS)$$
$$= \frac{1}{2}d\left[2(a - d\,\text{cosec}\,\theta) + 2d\cot\theta + 2(a - d\,\text{cosec}\,\theta)\right]$$
$$= \frac{1}{2}d\left[4a - 4(d\,\text{cosec}\,\theta) + 2d\cot\theta\right]$$
$$= 2ad + d^2(\cot\theta - 2\text{cosec}\,\theta) \quad \text{(shown)}$$

(b)

$\dfrac{\text{d}A}{\text{d}\theta} = -d^2\text{cosec}^2\theta + 2d^2\text{cosec}\,\theta\cot\theta$

When $\dfrac{\text{d}A}{\text{d}\theta} = 0$, $-d^2\text{cosec}^2\theta + 2d^2\text{cosec}\,\theta\cot\theta = 0$

$\text{cosec}\,\theta(\text{cosec}\,\theta - 2\cot\theta) = 0$

$\text{cosec}\,\theta = 0$ or $\text{cosec}\,\theta - 2\cot\theta = 0$

$\left(\text{rejected} \because \text{cosec}\,\theta = \dfrac{1}{\sin\theta} \neq 0 \text{ as } \theta > 0\right)$

$\therefore \text{cosec}\,\theta = 2\cot\theta$

$\cos\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}$

$\dfrac{\text{d}A}{\text{d}\theta} = -d^2\text{cosec}^2\theta + 2d^2\text{cosec}\,\theta\cot\theta$

θ	$\dfrac{\pi}{3}^-$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{3}^+$
$\dfrac{\text{d}A}{\text{d}\theta}$	>0	0	<0
Slope	/	–	\

Using the first derivative test, $\theta = \dfrac{\pi}{3}$ gives a maximum value for A.

Alternative method

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -d^2\operatorname{cosec}^2\theta + 2d^2\operatorname{cosec}\theta\cot\theta$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = 2d^2\operatorname{cosec}^2\theta\cot\theta - 2d^2\left(\operatorname{cosec}^3\theta + \operatorname{cosec}\theta\cot^2\theta\right)$$

At $\theta = \frac{\pi}{3}$, $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -\,2.309d^2 \quad > 0$

$\therefore \theta = \frac{\pi}{3}$ gives a maximum value for A.

At $\theta = \frac{\pi}{3}$, $A = 2ad + d^2\left(\cot\frac{\pi}{3} - 2\operatorname{cosec}\frac{\pi}{3}\right)$

$$= 2ad + d^2\left(\frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)\right)$$

$$= 2ad - d^2\sqrt{3}$$

13

(a) By Pythagoras' Theorem,

$r^2 + h^2 = 2^2$

$\therefore \ h = \sqrt{4-r^2}$ (reject $h = -\sqrt{4-r^2}$ since $h > 0$)

Let r_0 be radius of cut off cone.

Using similar $\triangle$s, $\frac{r_0}{r} = \frac{1}{4} \Rightarrow r_0 = \frac{r}{4}$

$$\therefore V = \pi r^2 (r) + \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_0{}^2 (\frac{1}{4}h)$$

$$= \pi r^3 + \frac{1}{3}\pi r^2\sqrt{4-r^2} - \frac{1}{192}\pi r^2\sqrt{4-r^2}$$

$$= \pi r^3 + \frac{21}{64}\pi r^2\sqrt{4-r^2} \quad \text{(shown)}$$

(b)
$$\frac{\mathrm{d}V}{\mathrm{d}r} = 3\pi r^2 + \frac{21}{64}\pi\left[2r\sqrt{4-r^2} + r^2(\frac{1}{2\sqrt{4-r^2}})(-2r)\right]$$

$$= 3\pi r^2 + \frac{21}{64}\pi r\left[2\sqrt{4-r^2} - \frac{r^2}{\sqrt{4-r^2}}\right]$$

$$= 3\pi r^2 + \frac{21}{64}\pi r\left[\frac{2\sqrt{4-r^2}\sqrt{4-r^2}}{\sqrt{4-r^2}} - \frac{r^2}{\sqrt{4-r^2}}\right]$$

$$= 3\pi r^2 + \frac{21}{64}\pi r\frac{8-3r^2}{\sqrt{4-r^2}}$$

At maximum value of V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$, since $0 < r < 2$

$$\therefore 3\pi r^2 + \frac{21}{64}\pi r\frac{8-3r^2}{\sqrt{4-r^2}} = 0$$

$$3\pi r\left[r + \frac{7}{64}\left(\frac{8-3r^2}{\sqrt{4-r^2}}\right)\right] = 0$$

$$r + \frac{7}{64}\left(\frac{8-3r^2}{\sqrt{4-r^2}}\right) = 0 \quad (\text{since } 0 < r < 2)$$

$$r = \frac{7}{64}\left(\frac{3r^2-8}{\sqrt{4-r^2}}\right)$$

$$64r\sqrt{4-r^2} = 21r^2 - 56$$

$$4096r^2(4-r^2) = (21r^2-56)^2$$

$$16384r^2 - 4096r^4 = 441r^4 - 2352r^2 + 3136$$

$$\therefore 4537r^4 - 18736r^2 + 3136 = 0 \quad \text{(shown)}$$

(c) Using GC, since $r > 0$,

$r = 0.41806129085388$ or $r = 1.9886743863834$

When $r = 0.41806129085388$, $\frac{\mathrm{d}V}{\mathrm{d}r} = 3.294435715 \neq 0$

Hence $r = 0.41806129085388$ does not give a stationary value of V.
The squaring of the equation to remove the square root created additional roots to the equation.
These additional roots may not give rise to stationary values of V.

(d) $r_1 = 1.99$ (3 s.f.)

$h = \sqrt{4 - r_1^2} = 0.2125421957 = 0.213$ (3 s.f.)

(e) The value of $r_1 = 1.99$ is almost the same length as the slant height 2 of the inverted cone. This means that the inverted cone is essentially flat and non-existent, leaving the printer nozzle in the shape of a cylinder only. Hence it is not realistic to have a maximum volume for the printer nozzle.

Ideal measurements/values/conditions based on theoretical calculations may not be practical or feasible in real life.

14

(a) Length of $BC = 2r\cos\theta$

Length of $CD = 2r\sin\theta$

$$P = 2[AB + BC + CD]$$
$$= 2[4r + 2r\cos\theta + 2r\sin\theta]$$
$$= 4r(2 + \cos\theta + \sin\theta) \quad \text{(shown)}$$

(b) $$\frac{dP}{d\theta} = 4r(-\sin\theta + \cos\theta)$$

At maximum P, $4r(-\sin\theta + \cos\theta) = 0$

$$\sin\theta = \cos\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

<u>Method ①:</u>

θ	$\left(\frac{\pi}{4}\right)^-$	$\frac{\pi}{4}$	$\left(\frac{\pi}{4}\right)^+$
$\frac{dP}{d\theta}$	> 0	0	< 0
	/	—	\

Hence, P is maximum when $\theta = \frac{\pi}{4}$.

<u>Method ②:</u>

$$\frac{d^2P}{d\theta^2} = 4r(-\cos\theta - \sin\theta)$$

When $\theta = \frac{\pi}{4}$,

$$\frac{d^2P}{d\theta^2} = 4r\left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) < 0$$

Hence, P is maximum when $\theta = \frac{\pi}{4}$.

$$\text{Maximum Distance} = 4r\left(2+\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)$$
$$= 4r\left(2+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)$$
$$= 4r\left(2+\sqrt{2}\right) \text{ metres}$$

(c) 3 minutes $= 180$ seconds

$$\text{Time to complete one loop} = \frac{4r\left(2+\sqrt{2}\right)}{6}$$

To find maximum value of r,

$$\frac{4r\left(2+\sqrt{2}\right)}{6} = 180$$
$$r = \frac{180\times 6}{4\left(2+\sqrt{2}\right)}$$
$$= \frac{270}{\left(2+\sqrt{2}\right)}\times\frac{2-\sqrt{2}}{2-\sqrt{2}}$$
$$= 135\left(2-\sqrt{2}\right)$$
$$= 270-135\sqrt{2}$$

(d)

$$\text{Area of } \Delta BCD = \frac{1}{2}(2r)(2r\cos\theta\sin\theta)$$
$$= 2r^2\cos\theta\sin\theta$$
$$\text{Area of } ABCDEF = (4r)(2r)+2\left(2r^2\cos\theta\sin\theta\right)$$
$$= 8r^2+4r^2\cos\theta\sin\theta$$

When $\theta = \frac{\pi}{4}$ and $r = 270-135\sqrt{2}$,

Cost of planting grass for *ABCDEF*

$$= 0.15\times\left[8\left(270-135\sqrt{2}\right)^2+4\left(270-135\sqrt{2}\right)^2\cos\frac{\pi}{4}\sin\frac{\pi}{4}\right]$$
$$= \$9380.75 < \$10000$$

Hence, management can afford to cover the entire shape *ABCDEF* with grass.

15

Let the volume of the pyramid by V.

$$h^2+\left(\frac{1}{2}x\right)^2=\left(\frac{1}{2}d\right)^2$$

$$x^2=d^2-4h^2$$

$$V=\frac{1}{3}x^2h$$

$$V=\frac{1}{3}h\left(d^2-4h^2\right)$$

$$=\frac{1}{3}d^2h-\frac{4}{3}h^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}d^2-4h^2$$

When $\frac{\mathrm{d}V}{\mathrm{d}h}=0$,

$$\frac{1}{3}d^2-4h^2=0$$

$$h^2=\frac{d^2}{12}$$

$$h=\sqrt{\frac{d^2}{12}}\ \text{(since } h>0\text{)}$$

$$=\frac{d}{2\sqrt{3}}$$

$$V=\frac{1}{3}x^2h$$

$$=\frac{1}{3}\left(d^2-\frac{4d^2}{12}\right)\left(\frac{d}{2\sqrt{3}}\right)$$

$$=\frac{d^3}{9\sqrt{3}}$$

Solution 2 (take *x* as the variable)

Let the volume of the pyramid by V.

Base area $= x^2$

$$h=\sqrt{\left(\frac{1}{2}d\right)^2-\left(\frac{1}{2}x\right)^2}$$
$$=\frac{1}{2}\sqrt{d^2-x^2}$$

$$V=\frac{1}{3}x^2h$$
$$=\frac{1}{3}x^2\left(\frac{1}{2}\sqrt{d^2-x^2}\right)$$
$$=\frac{1}{6}x^2\sqrt{d^2-x^2}$$
$$=\frac{1}{6}x^2\left(d^2-x^2\right)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{1}{6}x^2\left[\frac{1}{2}(-2x)\left(d^2-x^2\right)^{-\frac{1}{2}}\right]+\frac{1}{6}(2x)\left(d^2-x^2\right)^{\frac{1}{2}}$$
$$=-\frac{1}{6}x^3\left(d^2-x^2\right)^{-\frac{1}{2}}+\frac{1}{3}x\left(d^2-x^2\right)^{\frac{1}{2}}$$
$$=\frac{1}{6}x\left(d^2-x^2\right)^{-\frac{1}{2}}\left[-x^2+2\left(d^2-x^2\right)\right]$$
$$=\frac{x\left(2d^2-3x^2\right)}{6\sqrt{d^2-x^2}}$$

When $\frac{\mathrm{d}V}{\mathrm{d}x}=0$,

$x=0$ or $2d^2=3x^2$

$x=0$ or $x=\sqrt{\frac{2}{3}}d$ or $x=-\sqrt{\frac{2}{3}}d$

Since $0<x<d$, $x=\sqrt{\frac{2}{3}}d$.

$$V=\frac{1}{6}x^2\sqrt{d^2-x^2}$$
$$=\frac{1}{6}\left(\frac{2}{3}d^2\right)\sqrt{d^2-\frac{2}{3}d^2}$$
$$=\frac{1}{9}d^2\sqrt{\frac{1}{3}d^2}$$
$$=\frac{d^3}{9\sqrt{3}}$$

16

(i) $h^2 + r^2 = 8^2$

$r^2 = 64 - h^2$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(64 - h^2\right)h = \frac{1}{3}\pi\left(64h - h^3\right)$

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi\left(64 - 3h^2\right)$

At Stat point, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

$64 - 3h^2 = 0$

$h^2 = \frac{64}{3}$

$h = \frac{8\sqrt{3}}{3}$ (since $h > 0$)

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -2\pi h$

When $h = \frac{8\sqrt{3}}{3}, \frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -2\pi\left(\frac{8\sqrt{3}}{3}\right) = -\frac{16}{3}\sqrt{3}\pi < 0$

$\therefore h = \frac{8\sqrt{3}}{3}$ will give a maximum V

$\therefore V = \frac{1}{3}\pi\left[64\left(\frac{8\sqrt{3}}{3}\right) - \left(\frac{8\sqrt{3}}{3}\right)^3\right]$

$= \frac{1}{3}\pi\left(\frac{8\sqrt{3}}{3}\right)\left[64 - \left(\frac{8\sqrt{3}}{3}\right)^2\right]$

$= \pi\left(\frac{8\sqrt{3}}{9}\right)\left(64 - \left(\frac{8\sqrt{3}}{3}\right)^2\right)$

$= \frac{1024\sqrt{3}}{27}\pi$

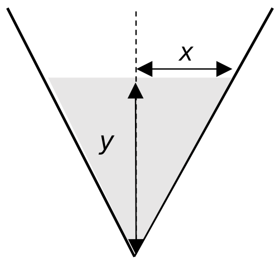
(ii) $r^2 = 64 - h^2 = 64 - \frac{64}{3} = \frac{128}{3}$

$\frac{h}{r} = \sqrt{\frac{64}{3}} \div \sqrt{\frac{128}{3}} = \frac{1}{\sqrt{2}}$

$r = \sqrt{2}h$ (shown)

(iii) Let the height and volume of the water be y cm and V' cm^3 respectively.

By similar triangle,



$$\frac{y}{x}=\frac{h}{r}$$

$$x=\sqrt{2}y$$

$$V'=\frac{1}{3}\pi x^2 y=\frac{1}{3}\pi\left(\sqrt{2}y\right)^2 y=\frac{2}{3}\pi y^3$$

$$\frac{\mathrm{d}V'}{\mathrm{d}y}=2\pi y^2$$

When $t=6,\quad V'=18\pi$

$$18\pi=\frac{2}{3}\pi y^3$$

$$y^3=27$$

$$y=3$$

$$\frac{\mathrm{d}V'}{\mathrm{d}t}=\frac{\mathrm{d}V'}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$$

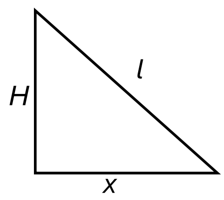
$$3\pi=2\pi(3)^2\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{6}\text{cm/s}$$

17

(a) $144 = 4 \times \frac{1}{2}(2x)(l) + (2x)^2$

$l = \frac{36}{x} - x$



Let H be the height of the square pyramid

$H = \sqrt{l^2 - x^2}$

$$V = \frac{1}{3}(2x)^2\sqrt{l^2 - x^2}$$
$$= \frac{1}{3}(2x)^2\sqrt{\left(\frac{36}{x} - x\right)^2 - x^2}$$
$$= \frac{1}{3}(4x^2)\sqrt{\left(\frac{36}{x}\right)\left(\frac{36}{x} - 2x\right)}$$
$$= \frac{1}{3}(4)(6)\sqrt{(x^2)^2\left(\frac{36}{x^2} - 2\right)}$$
$$= 8\sqrt{36x^2 - 2x^4}$$

(b) $$\frac{dV}{dx} = 8\left(\frac{1}{2}\right)\frac{1}{\sqrt{36x^2 - 2x^4}}(72x - 8x^3)$$
$$= \frac{32x(9 - x^2)}{\sqrt{36x^2 - 2x^4}}$$

At maximum V, $\frac{dV}{dx} = 0$.

$$\frac{32x(9 - x^2)}{\sqrt{36x^2 - 2x^4}} = 0$$

$x(3 - x)(3 + x) = 0$

$\Rightarrow x = 0$ or $x = \pm 3$

From context, $x > 0$. Hence $x = 3$.

x	3^-	3	3^+
$\frac{dV}{dx}$	> 0	0	< 0
Slope	/	–	\

Hence V is maximum when $x = 3$

Alternatively,

From GC, $\left.\frac{d^2V}{dx^2}\right|_{x=3} = -45.3 < 0$

Hence V is maximum when $x = 3$

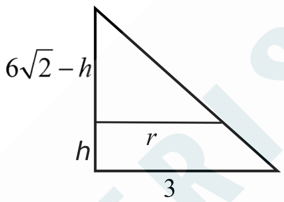
$$\text{Maximum } V = 8\sqrt{36(3)^2 - 2(3^4)}$$
$$= 8\sqrt{162}$$
$$= 8\sqrt{81(2)}$$
$$= 8(9)\sqrt{2}$$
$$= 72\sqrt{2} \text{ (Shown)}$$

(c) Let h, $2r$, W be the depth of liquid, side length of liquid surface, volume of liquid respectively.

From (a),

$$l = \frac{36}{3} - 3 = 9$$
$$H = \sqrt{l^2 - x^2} = \sqrt{9^2 - 3^2} = 6\sqrt{2}$$

By similar triangles, $\dfrac{r}{6\sqrt{2} - h} = \dfrac{3}{6\sqrt{2}}$



$$r = \frac{1}{2\sqrt{2}}\left(6\sqrt{2} - h\right)$$
$$W = 72\sqrt{2} - \frac{1}{3}(2r)^2\left(6\sqrt{2} - h\right)$$
$$= 72\sqrt{2} - \frac{1}{3}\left(\frac{1}{2}\right)\left(6\sqrt{2} - h\right)^3$$
$$= 72\sqrt{2} - \frac{1}{6}\left(6\sqrt{2} - h\right)^3$$

Given $\dfrac{\mathrm{d}W}{\mathrm{d}t} = 10$, $t = 6$,

$$10(6) = 72\sqrt{2} - \frac{1}{6}\left(6\sqrt{2} - h\right)^3$$
$h = 2.1778$ (5 sf)

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{6}(3)\left(6\sqrt{2} - h\right)^2(-1)\frac{\mathrm{d}h}{\mathrm{d}t}$$
$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{2}\left(6\sqrt{2} - h\right)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$

When $t = 6$,

$$10 = \frac{1}{2}\left(6\sqrt{2} - 2.1778\right)^2\frac{\mathrm{d}h}{\mathrm{d}t}$$
$\dfrac{\mathrm{d}h}{\mathrm{d}t} = 0.503$ (3 sf)

Depth is increasing at 0.503 cm per second

18

(a) Let $D = \mathrm{f}(x) - \mathrm{g}(x)$.

$$\frac{\mathrm{d}D}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f}(x) - \mathrm{g}(x)\right) = \mathrm{f}'(x) - \mathrm{g}'(x)$$

At $x = c$, distance is maximum, i.e. $\frac{\mathrm{d}D}{\mathrm{d}x} = 0$

$$\frac{\mathrm{d}D}{\mathrm{d}x} = 0 \Rightarrow \mathrm{f}'(c) - \mathrm{g}'(c) = 0 \Rightarrow \mathrm{f}'(c) = \mathrm{g}'(c) \text{ (shown)}$$

From the diagram, at $x = c$,
$y = \mathrm{f}(x)$ is concave downwards, i.e. $\mathrm{f}''(c) < 0$
$y = \mathrm{g}(x)$ is concave upwards, i.e. $\mathrm{g}''(c) > 0$

Therefore at $x = c$, $\frac{\mathrm{d}^2D}{\mathrm{d}x^2} = \mathrm{f}''(c) - \mathrm{g}''(c) < 0$.

Hence the distance is a maximum.

(b)(i)

(b)(ii) From (a), we know that the maximum will occur when $\mathrm{A}'(c) = \mathrm{B}'(c)$, and c will satisfy the equation $\mathrm{A}'(x) = \mathrm{B}'(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-0.16x^3 + 12x^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(5000\log_3\left(x+9\right) - 10\,000\right)$$

$$-0.48x^2 + 24x = \frac{5000}{\left(x+9\right)\ln 3}$$

Therefore $P = -0.48, Q = 24, R = \frac{5000}{\ln 3}$.

Using GC to solve the equation gives

$x = 11.906\,(5 \text{ s.f.}) = 11.9\,(3 \text{ s.f.})$

$x = 46.296\,(5 \text{ s.f.}) = 46.3\,(3 \text{ s.f.})$

$\therefore c = 11.9$ or $c = 46.3$

(b)(iii) $\mathrm{A}(11.906) - \mathrm{B}(11.906) = -2404.8$

$\mathrm{A}(46.296) - \mathrm{B}(46.296) = 1580.9$

Therefore the furthest distance between them is 2404.8m (or 2400m) and Ben is in front of Andy.

19

(a)

$$V = 64\left(he^{kt}+4\right)^{-1}$$

$$\frac{dV}{dt} = -64\left(he^{kt}+4\right)^{-2}\left(khe^{kt}\right)$$

$$= -64hke^{kt}\left(he^{kt}+4\right)^{-2}$$

$$\frac{d^2V}{dt^2} = -64hk\left[ke^{kt}\left(he^{kt}+4\right)^{-2} - 2e^{kt}\left(hke^{kt}\right)\left(he^{kt}+4\right)^{-3}\right]$$

$$= -64hk^2e^{kt}\left(he^{kt}+4\right)^{-3}\left(he^{kt}+4-2he^{kt}\right)$$

$$= -64hk^2e^{kt}\left(he^{kt}+4\right)^{-3}\left(4-he^{kt}\right)$$

When $\frac{d^2V}{dt^2} = 0$ and $t = t_1$,

$$-64hk^2e^{kt_1}\left(he^{kt_1}+4\right)^{-3}\left(4-he^{kt_1}\right) = 0$$

$4 - he^{kt_1} = 0$ since h, k and $e^{kt_1} > 0$

$t = t_1$ satisfies $he^{kt} - 4 = 0 \Rightarrow he^{kt} = 4$

(b)

When $\frac{dV}{dt}$ attains its minimum value, $he^{kt} = 4$.

$$V = \frac{64}{he^{kt}+4} = \frac{64}{4+4} = 8 \text{ m}^3$$

Differentiate $S = V^{\frac{2}{3}}$ w.r.t. t and let $he^{kt} = 4$ and $V = 8$

$$\frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$$

$$= \frac{2}{3}V^{-\frac{1}{3}} \times \frac{dV}{dt}$$

$$= \frac{2}{3}(8)^{-\frac{1}{3}} \times \left[-64k(4)(4+4)^{-2}\right]$$

$$= -\frac{4k}{3}$$

(c)

When $\frac{dV}{dt}$ attains its least value, $\frac{d^2V}{dt^2} = 0$ and $he^{kt} = 4$

Differentiate $\frac{dS}{dt} = \frac{2}{3}V^{-\frac{1}{3}}\left(\frac{dV}{dt}\right)$ w.r.t. t and let $\frac{d^2V}{dt^2} = 0$ and $he^{kt} = 4$,

$$\frac{d^2S}{dt^2} = \frac{2}{3}\left(-\frac{1}{3}\right)V^{-\frac{4}{3}}\left(\frac{dV}{dt}\right)^2 + \frac{2}{3}V^{-\frac{1}{3}}\left(\frac{d^2V}{dt^2}\right)$$

$$= \frac{2}{3}\left(-\frac{1}{3}\right)(8)^{-\frac{4}{3}}\left[-64k(4)(4+4)^{-2}\right]^2$$

$$= -\frac{2k}{9} \neq 0 \text{ since } k > 0$$

Hence, $\frac{dS}{dt}$ did not attain its least value when $\frac{dV}{dt}$ attains its least value.

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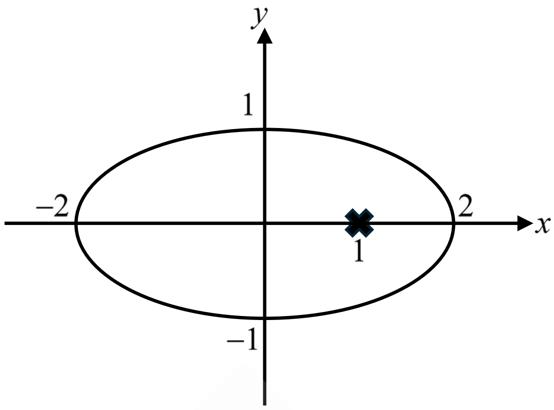
(i) A is at $(1,0)$.

B is at (x,y) where $y^2 = 1-\dfrac{x^2}{4}$



$$r^2 = (x-1)^2 + (y-0)^2$$
$$= (x-1)^2 + 1 - \frac{x^2}{4}$$
$$= \frac{1}{4}\left(3x^2-8x+8\right)$$

$$F = \frac{4k}{3x^2-8x+8}$$

(ii)
$$F = \frac{4k}{3x^2-8x+8}$$
$$\frac{\mathrm{d}F}{\mathrm{d}x} = -4k\left(3x^2-8x+8\right)^{-2}(6x-8)$$
$$= \frac{-8k(3x-4)}{\left(3x^2-8x+8\right)^2}$$

At stationary point, $\dfrac{\mathrm{d}F}{\mathrm{d}x} = 0$.

$$6x-8=0$$
$$x = \frac{4}{3}$$

Method 1: Using 2nd Derivative to verify nature

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2} = \frac{\left(3x^2-8x+8\right)^2(-24k)+8k(3x-4)(2)\left(3x^2-8x+8\right)(6x-8)}{\left(3x^2-8x+8\right)^4}$$

When $6x-8=0$, i.e. $x=\dfrac{4}{3}$:

$\dfrac{\mathrm{d}^2F}{\mathrm{d}x^2} = -3.375k < 0$, therefore maximum.

Note: $\left.\dfrac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=\frac{4}{3}}$ can be found easily using GC.

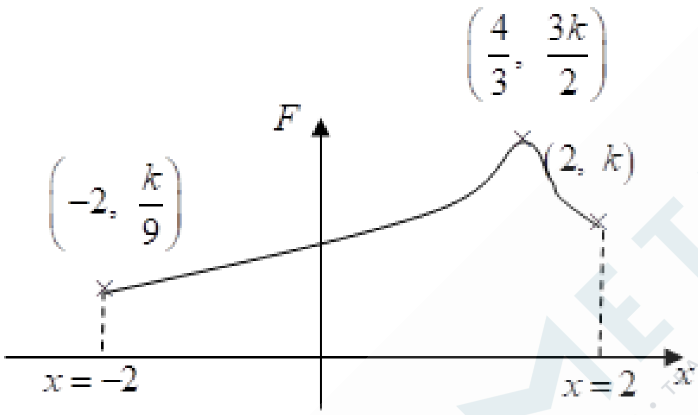
Method 2: Using 1st Derivative to verify nature

x	$\frac{4}{3}^-$	$\frac{4}{3}$	$\frac{4}{3}^+$
$\frac{dF}{dx}$	>0	0	<0
	/	-	\

$$F_{max} = \frac{4k}{3\left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 8}$$

$$= \frac{3k}{2}$$

(iii)



Note that $y^2 = 1 - \frac{x^2}{4} \Rightarrow x = \pm 2\sqrt{1-y^2}$

Therefore $-2 \le x \le 2$

(iv) Minimum F occurs when B is farthest from A.
i.e. when B is $(-2,\ 0)$, and thus $r = 3$.

$$F_{min} = \frac{k}{3^2}$$

$$F_{min} = \frac{k}{9}$$

(v) By symmetry, B must be at $(0,1)$ or $(0,-1)$ with x-coordinate $= 0$

With $x = 0$, $F = \frac{4k}{0+8} = \frac{k}{2}$