1

(a)

$$\frac{n}{2}\left[2(5)+(n-1)(3)\right]\le 100$$

$$\frac{n}{2}\left[2(5)+(n-1)(3)\right]-100\le 0$$

Using GC,

n	$\frac{n}{2}\left[2(5)+(n-1)(3)\right]-100$
6	-25
7	-2
8	24

Maximum number of squares Student A can form using the 100 cm wire is 7.

(b) The circumference of the circles follow a geometric progression with common ratio $\frac{2}{3}$.

$100 =$ Total circumference of 12 circles

$$100 = 2\pi x+\frac{2}{3}(2\pi x)+\left(\frac{2}{3}\right)^{2}(2\pi x)+\ldots+\left(\frac{2}{3}\right)^{11}(2\pi x)$$

$$100=\frac{2\pi x\left(1-\left(\frac{2}{3}\right)^{12}\right)}{1-\frac{2}{3}}$$

Using GC,

$x = 5.3464$

$= 5.35 \ (3\text{ s.f.})$

2

(a)

$$_n = S_n - S_{n-1}$$
$$= \left(n^2 + n\right) - \left[\left(n-1\right)^2 + \left(n-1\right)\right]$$
$$= \left(n^2 - (n-1)^2\right) + \left(n - (n-1)\right)$$
$$= \left(n + (n-1)\right)(n - (n-1)) + 1$$
$$= 2n - 1 + 1$$
$$= 2n$$

The general term $u_n = 2n$

$$u_n - u_{n-1} = 2n - \left(2(n-1)\right)$$
$$= 2 \ \left(\text{constant}\right)$$

Since $u_n - u_{n-1}$ is a constant independent of n, hence $\{u_n\}$ forms a GP.

(b) Let a denote the first term of the geometric progression.
Let b and d denote the first term and common difference of the arithmetic progression.

$\therefore$ $ar^2 = b + 6d$ …(1)
$ar^4 = b + 12d$ …(2)
$ar^6 = b + 24d$ …(3)

(2) – (1): $ar^4 - ar^2 = 6d$ …(4)
(3) – (2): $ar^6 - ar^4 = 12d$ …(5)

(4)/(5): $$\frac{ar^2\left(r^2-1\right)}{ar^4\left(r^2-1\right)} = \frac{6d}{12d}$$
$$\frac{r^2}{r^4} = \frac{1}{2}$$
$$\frac{1}{r^2} = \frac{1}{2}$$
$$r = \pm\sqrt{2}$$

Since $r > 0$, $r = \sqrt{2}$

Since $|r| > 1$, the geometric progression is not convergent.

3

(a) First term of AP: $a = 500$

Common difference of AP: $d = 10$

Formulation of problem: $\frac{n}{2}\left[2(500)+10(n-1)\right] > 10000$

Using GC,

n	$\frac{n}{2}\left[2(500)+10(n-1)\right]$
17	9860
18	10530
19	11210

$n = 18$

Date of 18th month: 1 June 2024

(b) Formulation of problem:

Month n	Start of month	End of month
1	x	$1.005x$
2	$1.005x + x$	$1.005^2 x + 1.005x$
3	$1.005^2 x + 1.005x + x$	$1.005^3 x + 1.005^2 x + 1.005x$
$\vdots$	$\vdots$	$\vdots$

At the end of Nth month, account has

$$\begin{aligned} 1.005^N x + 1.005^{N-1} x + \ldots + 1.005x &= x(1.005^N + 1.005^{N-1} + \ldots + 1.005) \\ &= x\left[\frac{1.005(1.005^N - 1)}{1.005 - 1}\right] \\ &= 201x(1.005^N - 1) \end{aligned}$$

Thus we have $N = 60$ at the end of 31 December 2027

$201x(1.005^{60} - 1) \geq 50{,}000 \quad \Rightarrow \quad x \geq \$713.0747 \quad \Rightarrow \quad \text{Least } x = \$714$

4

(a) $S_n = 3n(n+2)$

$u_n = S_n - S_{n-1}$

$= 3n(n+2) - 3(n-1)(n+1)$

$= 6n+3$

$u_n - u_{n-1} = 6n+3-\left(6(n-1)+3\right)$

$= 6n+3-6n+3$

$= 6 \ (\text{constant})$

Since the difference between two consecutive terms is a constant, the series is an arithmetic progression.

The common difference is 6.

(b) $v_1 = u_2 = 6(2)+3 = 15$

$_2 = 6(7)+3 = 45$

common ratio, $r = \dfrac{45}{15} = 3$

$_3 = 45(3) = 135$

The m^{th} term of the series in (i),

$135 = 6(m)+3$

$m = \dfrac{135-3}{6} = 22$

Since $r = 3$ does not lie within $-1 < r < 1$, the sum to infinity of v_n does not exist.

(c) common ratio $= \dfrac{w_n}{w_{n-1}}$

$= \dfrac{e^{5+nx(x+1)}}{e^{5+(n-1)x(x+1)}}$

$= \dfrac{e^5 e^{nx(x+1)}}{e^5 e^{(n-1)x(x+1)}}$

$= e^{nx(x+1)-(n-1)x(x+1)}$

$= e^{(x+1)[nx-(nx-x)]}$

$= e^{x(x+1)}$

For the series to converge, $\left|e^{x(x+1)}\right| < 1$, $x(x+1) < 0$

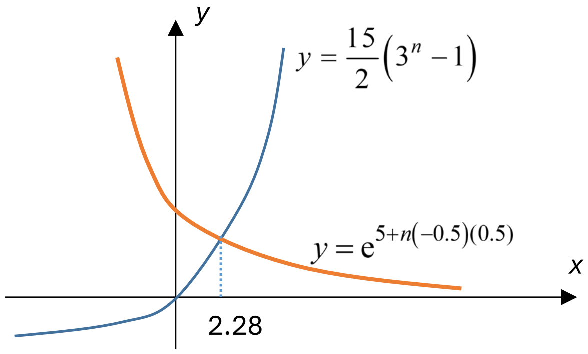
The range of values of x is $-1 < x < 0$.

(d) Sum of first n terms of v_n, S_{v_n}

$$=\frac{15\left(3^n-1\right)}{3-1}=\frac{15}{2}\left(3^n-1\right)$$

$S_{v_n} > w_n$ using $x=-0.5$,

$$\frac{15}{2}\left(3^n-1\right)>\mathrm{e}^{5+n(-0.5)(0.5)}$$



From the graph, the least value of n is 3.

Alternative (table method)

$$\frac{15}{2}\left(3^n-1\right)-\mathrm{e}^{5+n(-0.5)(0.5)}>0$$

From the GC,

n	$\frac{15}{2}\left(3^n-1\right)-\mathrm{e}^{5+n(-0.5)(0.5)}$
2	$-30.017<0$
3	$124.895>0$
4	$545.402>0$

The least value of n is 3.

5

(i) For the sprinter: 56 + 62 + 68 + … + $[56+(n-1)(6)]$

AP: First term = 56, common difference = 6

The time the sprinter takes to complete n laps $=\frac{n}{2}\left[2(56)+(n-1)(6)\right]$

$= n\left(56+3n-3\right)=3n^2+53n$ (Shown)

(ii) For the marathon runner: 60 + $60(1.04)+60(1.04)^2+60(1.04)^3+...+60(1.04)^{n-1}$

GP: First term = 60, common ratio = 1.04

The time the marathon runner takes to complete n laps = $\frac{60\left(1-1.04^n\right)}{1-1.04}$ or $\frac{60\left(1.04^n-1\right)}{1.04-1}$

$=1500\left(1.04^n-1\right)$

(iii) Consider $1500\left(1.04^n-1\right)>2400$

Method 1

$1.04^n-1>1.6$

$1.04^n>2.6$

Taking lg on both sides: $n>\frac{\lg 2.6}{\lg 1.04}$

$\therefore\ n>24.362$ (correct to 5 s.f.)

The marathon runner needs to complete 25 laps.

Method 2

n	$1500\left(1.04^n-1\right)$	
24	2345	<2400
25	2498.8	>2400
26	2658.7	>2400

The marathon runner needs to complete 25 laps.

(iv) $n=\frac{12000}{400}=30$

Time taken by the sprinter = $3(30)^2+53(30)=4290$ seconds

Time taken by the marathon runner = $1500\left(1.04^{30}-1\right)=3365.1$ seconds (correct to 5 s.f.)

The marathon runner will be the first to complete the race.

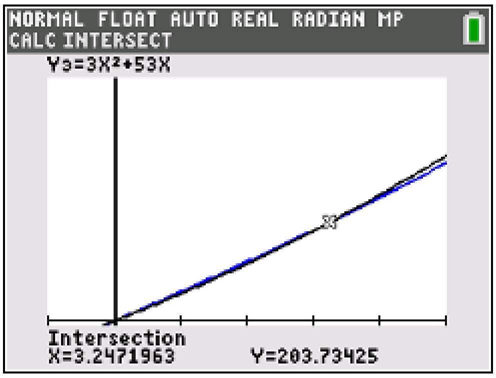
(v) Consider $1500\left(1.04^{n}-1\right)<3n^{2}+53n$

Method 1

n	$1500\left(1.04^{n}-1\right)$	$3n^{2}+53n$
3	187.3	186
4	254.79	260
5	324.98	340

Method 2

(Sketch $Y_1=1500\left(1.04^{X}-1\right)$, $Y_2=3X^{2}+53X$)



Since $n\geq 3.25$ (3 s.f.), $\therefore$ $n=4$

Hence, the marathon runner first overtakes the sprinter on his 4th lap.

6

(i)

Month n	Amount in account at the start of nth month	Amount in account at the end of nth month
1 (Jan22)	500	(1.001)(500)
2	(1.001)(500)+500	$1.001[1.001(500)+500]$ $=1.001^2(500)+1.001(500)$
12	…	$1.001^{12}(500)+$ $1.001^{11}(500)+..+1.001(500)$

Total amount at the end of 12 months

$$=1.001^{12}(500)+1.001^{11}(500)+..+1.001(500)$$

$$=500\underbrace{\left(1.001+1.001^2+...+1.001^{12}\right)}_{\text{GP: }a=1.001, r=1.001, n=12}$$

$$=500\left[\frac{1.001\left(1-1.001^{12}\right)}{1-1.001}\right]$$

$$=6039.1434$$

$$=6039.14$$

total amount of money: \$6039.14

(ii) Total amount at the end of n months

$$=1.001^n(500)+1.001^{n-1}(500)+..+1.001(500)$$

$$=500\underbrace{\left(1.001+1.001^2+...+1.001^{n}\right)}_{\text{GP: }a=1.001, r=1.001, n\text{ terms}}$$

$$=500\left[\frac{1.001\left(1-1.001^{n}\right)}{1-1.001}\right]$$

$$=-500500\left(1-1.001^n\right)$$

$$=500500\left(1.001^n-1\right)$$

For $500500\left(1.001^n-1\right)>20\ 000$:

Method 1

n	$500500\left(1.003^n-1\right)$
39	19895
40	20415
41	20936

Minimum number of months is 40.

Method 2

$$500500\left(1.001^n-1\right)>20\ 000$$

$$1.001^n>\frac{1041}{1001}$$

$$n\ln 1.001>\ln\left(\frac{1041}{1001}\right)$$

$$n>39.2$$

Minimum number of months is 40.

(iii)

Month m	Amount in account at the start of mth month	Amount in account at the end of mth month
1 (Jan22)	50 000	$1.001(50000)-k$
2	$1.001(50000)-k$	$1.001(1.001(50000)-k)-k$ $=1.001^2(50000)-1.001k-k$
3	$1.001^2(50000)-1.001k-k$	$1.001(1.001^2(50000)-1.001k-k)-k$ $=1.001^3(50000)-(1.001)^2k-1.001k-k$

Amount of money in account at the end of mth month

$$=1.001^m(50000)-(1.001)^{m-1}k-\ldots-1.001k-k$$

$$=1.001^m(50000)-k\underbrace{\left(1+1.001+\ldots+1.001^{m-1}\right)}_{\text{GP: } a=1, r=1.001, m \text{ terms}}$$

$$=1.001^m(50000)-k\left(\frac{1\left(1-1.001^m\right)}{1-1.001}\right)$$

$$=1.001^m(50000)+1000k\left(1-1.001^m\right)$$

$$=1.001^m\left[50000-1000k\right]+1000k \text{ (shown)}$$

(iv) At the end of 2024 refers to a duration of 3 years.
let $m = 36$,

$$1.001^{36}\left[50000-1000k\right]+1000k=0$$

$$k\left[1.001^{36}(1000)-1000\right]=1.001^{36}(50000)$$

Using GC,
$k=1414.73$

Maximum amount of money is \$1414.

7

(a)(i) First term of the arithmetic progression $= ar$

$$a = \frac{4}{2}\left[2(ar) + 3(2d)\right]$$

$$= 4ar + 12d$$

$$a = 4\left(ar + 3d\right) \quad \text{-- (Eq.1)}$$

$$ar^2 = ar + 3d \qquad \text{-- (Eq.2)}$$

Take $\frac{\text{Eq.2}}{\text{Eq.1}}$, we have

$$\frac{ar^2}{a} = \frac{ar + 3d}{4\left(ar + 3d\right)}$$

$$r^2 = \frac{1}{4} \quad \text{(shown)}$$

Alternative Solution

$$a = 4b + 12d \quad \text{-- (Eq.3)}$$

$$ar^2 = b + 3d \qquad \text{-- (Eq.4)}$$

Take $\frac{\text{Eq.4}}{\text{Eq.3}}$, we have

$$\frac{ar^2}{a} = \frac{b + 3d}{4\left(b + 3d\right)}$$

$$r^2 = \frac{1}{4} \quad \text{(shown)}$$

(a)(ii) **<u>Method 1</u>**

$$r^2 = \frac{1}{4} \Rightarrow r = \frac{1}{2} \quad \text{or} \quad -\frac{1}{2}$$

Since $S_\infty = \frac{a}{1-r} < a$, we have

$$\frac{a}{1-r} < a$$

$$1 - r > 1 \ (\because a > 0 \text{ and } 1 - r > 0)$$

$$r < 0$$

Thus, $r = -\frac{1}{2}$

<u>Method 2</u>

When $r = \frac{1}{2}$, $S_\infty = \frac{a}{1-\left(\frac{1}{2}\right)} = 2a \not< a$

When $r = \frac{1}{2}$, $S_\infty = \frac{a}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}a < a$

Thus, $r = -\frac{1}{2}$.

<u>Method 3</u>

In order for sum to infinity to be less than a, there must be some negative terms after the first. If all the terms had been positive, the sum of the terms will never be less than a. Since the first term is positive, then r has to be negative in order to create these negative terms. (If r had been positive, then all the terms will be positive.) Hence, $r = -\frac{1}{2}$.

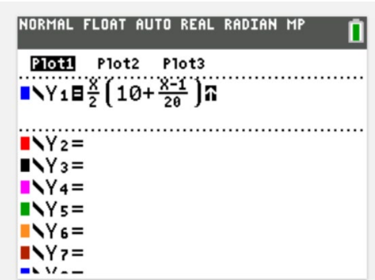
(b) $u_{25} = a + 24d = \dfrac{31}{5}$

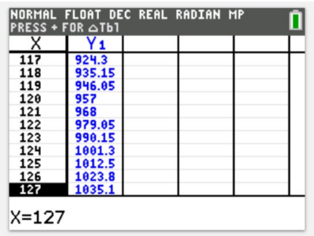
$$24d = \frac{31}{5} - 5 = \frac{6}{5}$$

$$d = \frac{1}{20}$$

$$S_n > 1000$$

$$\frac{n}{2}[2(5) + (n-1)\frac{1}{20}] > 1000$$





From GC, $n = 124$

8

(a)

nth day	Number of daily views at the end of nth day
1	1196
2	$3(1196)$
3	$3^2(1196)$

G.P. with first term = 1196 and common ratio = 3

Number of daily views at the end of the third day

$=3^2(1196)$

$=10764$

(b) Total number of views at the end of the 7th day

$$=\frac{1196(3^7-1)}{3-1}$$

$=1307228$

$<5000\ 000$

The video will not go viral.

(c)

$$\frac{n}{2}\left[2(576)+(n-1)(780)\right]>100\ 000$$

$$576n+390n^2-390n>100\ 000$$

$$390n^2+186n-100\ 000>0$$

Using G.C.,

$n<-16.253$ or $n>15.776$

OR

n	$390n^2+186n-100\ 000$	
15	-9460	<0
16	2816	>0
17	15872	>0

Least $n=16$

(d) Total number of comments at the end of Day 16

$= \frac{16}{2}\left[2(576)+(16-1)(780)\right]$

$= 102816$

OR

$= 100\ 000 + 2816$ (from G.C. table)

$= 102816$

n	Start of Day	End of Day
1	$102\ 816 - w$	$1.03(102\ 816 - w)$ $= 1.03(102\ 816) - 1.03w$
2	$1.03(102\ 816) - 1.03w - w$	$1.03\left[1.03(102\ 816) - 1.03w - w\right]$ $= (1.03)^2(102\ 816) - (1.03)^2 w - 1.03w$
	$(1.03)^2(102\ 816) - (1.03)^2 w - 1.03w - w$	$(1.03)^3(102\ 816) - (1.03)^3 w - (1.03)^2 w - 1.03w$

Number of comments by the end of Day n

$$(1.03)^n(102\ 816) - (1.03)^n w - (1.03)^{n-1} w - \ldots - 1.03w$$

$$(1.03)^n(102\ 816) - \left[(1.03)^n w + (1.03)^{n-1} w + \ldots + 1.03w\right]$$

$$(1.03)^n(102\ 816) - \frac{1.03w\left[(1.03)^n - 1\right]}{1.03 - 1}$$

$$(1.03)^n(102\ 816) - \frac{103w}{3}\left[(1.03)^n - 1\right] \text{ (shown)}$$

where $M = 102816$

(e)

$$(1.03)^{31}(102\ 816) - \frac{103w}{3}\left[(1.03)^{31} - 1\right] \le 0$$

$$\frac{103w}{3}\left[(1.03)^{31} - 1\right] \ge (1.03)^{31}(102\ 816)$$

$$w \ge 4990.961031$$

$$w \ge 4991$$

9

(i)(a) 700, 700+60, 700+60+60, …

This is an AP with a = 700 and d = 60

Hence the total paid after the k^{th} payment = $\frac{k}{2}\left[2(700)+(k-1)60\right]$

$= k(700+30k-30)$

$= 30k^2+670k$

$= \$(30k^2+670k)$ (Shown)

(i)(b) $30k^2+670k \geq 40000 + 4660$

$30k^2+670k-44660 \geq 0$

From the GC,

k	$30k^2 + 670k - 44660$
28	-2380
29	0
30	2440

$\therefore$ It will take Mr Kim 29 payments to fully repay his loan.

(ii)(a)

Month	Amount owed at the end of the month
Jan	$40000(1.015) - p$
Feb	$[40000(1.015) - p](1.015) - p$

$\therefore$ The amount he owes on 1st Mar 2023 = $40000(1.015)^2 - 1.015p - p$

$= 41209 - 2.015p$

(ii)(b)

End of	Amount owed after interest	Amount owed after payment
Jan $n=1$	$40000(1.015)$	$40000(1.015)-p$
Feb $n=2$	$40000(1.015)^2-p(1.015)$	$40000(1.015)^2-p(1.015)-p$
Mar $n=3$	$40000(1.015)^3-p(1.015)^2$ $-p(1.015)$	$40000(1.015)^3-p(1.015)^2$ $-p(1.015)-p$
…	…	…
n		$40000(1.015)^n$ $-p(1.015)^{n-1}-p(1.015)^{n-2}$ $-\cdots-p(1.015)-p$

The amount he owed at the start of the nth month (is the amount he owed at the end of the n^{th} month after interest is charged and \$$p$ payment is made)

$= 40000(1.015)^n - p(1.015)^{n-1} - p(1.015)^{n-2} - \cdots - p(1.015) - p$

$= 40000(1.015)^n - p\left[(1.015)^{n-1} + (1.015)^{n-2} + \cdots + (1.015)^1 + 1\right]$

$= 40000(1.015)^n - p\left[\frac{1.015^n - 1}{1.015 - 1}\right]$

$= 40000(1.015)^n - \frac{200}{3}p\left(1.015^n - 1\right)$, where $\alpha = 1.015$ and $\beta = \frac{200}{3}$

(ii)(c)

$$40000(1.015)^n - \frac{200}{3}(1585)\left(1.015^n - 1\right) \ge 0$$

$$197(1.015)^n \le 317$$

$$n\ln(1.015) \le \ln\frac{317}{197}$$

$n \le 31.95046 \quad \Rightarrow$ Mr Kim still owes money up to the 31st payment.

$\therefore$ Mr Kim will fully pay off his loan in 32 months, i.e. $k = 32$.

Under plan B, amount that Mr Kim owes at the end of 31st month after interest and payment

$$= 40000(1.015)^{31} - \frac{200}{3}(1585)\left(1.015^{31} - 1\right)$$

$= 1484.764837$

Amount that Mr Kim needs to pay at the end of the 32nd month $= 1484.764837 \times 1.015$

$= \$1507.04$

Mr Kim will pay \$1507.04 at the end of the 32nd month after interest, i.e. $m = 1507.04$

10

(a) the amount of money Jeremy will have at the end of the year 2025 is

$$100000\times1.032+X)\times1.032=100000\times1.032^2+1.032X$$
$$=106502.40+1.032X$$

The amount is $\$(106502.40+1.032X)$.

(b)

Year	Amount at start of year	Amount at end of year
2024 $n=1$	100000	$100000(1.032)$
2025 $n=2$	$100000(1.032)+X$	$[100000(1.032)+X](1.032)$ $=1.032^2(100000)+1.032X$
2026 $n=3$	$1.032^2(100000)+1.032X+X$	$1.032^3(100000)+1.032^2X+1.032X$
…	…	…
N		$1.032^n(100000)+1.032^{n-1}X+1.032^{n-1}X+\ldots+1.032X$

$$100000\left(1.032^n\right)+\left(1.032^{n-1}\right)X+\left(1.032^{n-2}\right)X+\ldots+1.032X$$
$$=100000\left(1.032^n\right)+\frac{(1.032)X\left(1-1.032^{n-1}\right)}{1-1.032}$$
$$=100000\left(1.032^n\right)+32.25X\left(1.032^{n-1}-1\right)$$

The amount of money Jeremy will have at the end of n years is
$\$\left[100000\left(1.032^n\right)+32.25X\left(1.032^{n-1}-1\right)\right]$

(c) To fully pay the car loan

$$100000\left(1.032^n\right)+32.25(8000)\left(1.032^{n-1}-1\right)\geq170000$$
$$100000\left(1.032^n\right)+258000\left(1.032^{n-1}-1\right)\geq170000$$

Using GC,

n	Amount
6	164811
7	178341
8	192304

$n=7$
$164811+8000=172811>170000$

Hence he will have sufficient amount in the account on the first day of 2030.

(d) Total amount $= 170000 + 170000 \times 0.028m$

$= 170000 + 4760m$

Thus the annual instalment is $\$\frac{1}{m}(170000 + 4760m)$.

When $m = 10$, the annual instalment is $\frac{1}{10}(170000 + 4760 \times 10) = 21760$, he needs to withdraw $21760 - 8000 = 13760$ from his current account every year.

Take $X = -13760$ for **(b)** answer, the amount in the current account at the end of n years is

$100000(1.032^n) + 32.25(-13760)(1.032^{n-1} - 1)$

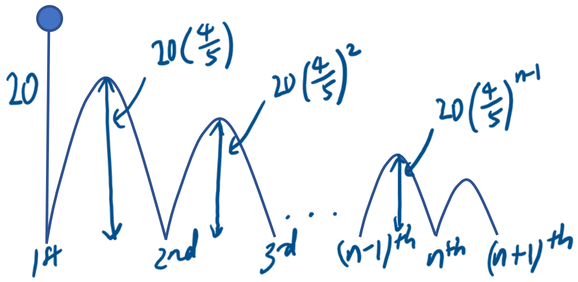
$= 100000(1.032^n) - 443760(1.032^{n-1} - 1)$

By GC,

n	Amount in the current account
9	5601.5
10	– 8420

Jeremy will not have sufficient money in his current account at the end of 10^{th} year, so the arrangement is not possible.

11



Distance travelled at the n^{th} bounce

$$=20+20(2)\left[\frac{4}{5}+\left(\frac{4}{5}\right)^2+\left(\frac{4}{5}\right)^3+\ldots+\left(\frac{4}{5}\right)^{n-1}\right]$$

$$=20+40\left[\frac{\frac{4}{5}\left(1-\left(\frac{4}{5}\right)^{n-1}\right)}{1-\frac{4}{5}}\right]$$

$$=20+160\left(1-\left(\frac{4}{5}\right)^{n-1}\right)$$

$$=180-160\left(\frac{4}{5}\right)^{n-1} \quad \left(\text{or } 180-200\left(\frac{4}{5}\right)^{n}\right)$$

n	$180-160\left(\frac{4}{5}\right)^{n-1}$
8	146.445568
9	153.156454
10	158.525164

Therefore, when the ball has travelled exactly 148m, it has only bounced 8 times.

OR

$$180-160\left(\frac{4}{5}\right)^{n-1} \geq 148$$

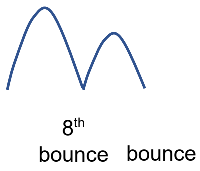
$$160\left(\frac{4}{5}\right)^{n-1} \leq 32$$

$$(n-1)\ln\left(\frac{4}{5}\right) \leq \ln\frac{32}{160}$$

$$n \geq 8.2126$$

Therefore, when the ball has travelled exactly 148m, it has only bounced 8 times.

(a) Dist. travelled btw 8^{th} and 9^{th} bounce = 153.1564544 – 146.445568
= 6.7108864 m



Max height is exactly $\frac{6.7108864}{2} = 3.3554432$ m above the platform.

$k = 3.3554432 - (148 - 146.445568) = 1.8010112 = 1.80$ (2 dp)

OR

Refer to diagram in earlier part of question,

Max height is exactly $20\left(\frac{4}{5}\right)^8 = 3.3554432$ m above the platform.

$$k = 3.3554432 - [148 - \left(180 - 160\left(\frac{4}{5}\right)^{8-1}\right)] = 1.8010112$$

$$= 1.80 \text{ (2 dp)}$$

(b) After the 8^{th} bounce, ball reaches max height of 3.3554432 m

New height *h'* on lowered platform $3.3554432 + 2 = 5.3554432$ m

Distance travelled on the lowered platform after experiment resumes

$$= k + h' + 2h'\left[\frac{4}{5} + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \ldots\right]$$

$$= 1.8010112 + h' + 2h'\left[\frac{\frac{4}{5}}{1-\frac{4}{5}}\right]$$

= 50 m

12

(a) $u_{n+1} = (1+p)u_n$

Since $\dfrac{u_{n+1}}{u_n} = 1+p = \text{constant}$ (since p is a constant),

$\therefore$ $\{u_n\}$ is a geometric progression.

(b)

n	u_n
1	2^m
2	$(1+p)u_1 = (1+p)2^m$
3	$(1+p)u_2 = (1+p)[(1+p)2^m]$ $= (1+p)^2 2^m$

$$2^m + (1+p)2^m + (1+p)^2 2^m = \frac{127}{36}(2^m)$$
$$1 + (1+p) + (1+2p+p^2) = \frac{127}{36}$$
$$p^2 + 3p + 3 = \frac{127}{36}$$
$$p^2 + 3p - \frac{19}{36} = 0$$
$$36p^2 + 108p - 19 = 0$$

Using GC,

$p = \dfrac{1}{6}$ or $p = -\dfrac{19}{6}$ (rejected since $p > 0$)

Hence $p = \dfrac{1}{6}$.

(c) Since common ratio $r = 1+p = \dfrac{7}{6} > 1$, sum to infinity does not exist. As $n \to \infty$, total amount of data $S_n = \dfrac{a(\frac{7}{6}^n - 1)}{\frac{7}{6} - 1}$ diverges. Hence there is no limit to total amount of data the data processing centre can handle.

(d)

n	v_n	
1	2^m	
2	$2^m(\frac{5}{4})$	
3	$2^m(\frac{5}{4})^2$	
4	$2^m(\frac{5}{4})^3$	AP with 1st term $2^m(\frac{5}{4})^3$, common difference -25
5	$2^m(\frac{5}{4})^3 - 25$	
6	$2^m(\frac{5}{4})^3 - 25(2)$	
$\vdots$		

From v_4 to v_r, no. of terms $= r-4+1 = r-3$

$$\therefore\ v_r = 2^m\left(\frac{5}{4}\right)^3 + [(r-3)-1](-25)$$

For AP: $T_n = a + (n-1)d$

$$= \frac{125}{2^6}(2^m) - 25(r-4)$$
$$= 125(2^{m-6}) - 25r + 100$$
$$= 25[5(2^{m-6}) - r + 4] \quad \text{where } k = 25 \quad \text{(shown)}$$

(e) $v_1 + v_2 + v_3 + \ldots + v_{15}$

$$= (2^m) + \frac{5}{4}(2^m) + \frac{25}{16}(2^m) + v_4 + \ldots\ldots + v_{15}$$

For AP: $S_n = \frac{n}{2}(a + T_n)$

$$= \left(1 + \frac{5}{4} + \frac{25}{16}\right)(2^m) + \frac{15-4+1}{2}(v_4 + v_{15})$$
$$= \frac{61}{16}(2^m) + 6\left[\frac{125}{64}(2^m) + \left(25[5(2^{m-6}) - 15 + 4]\right)\right]$$

$v_4 = \left(\frac{5}{4}\right)^3(2^m)$.
Replace r in part **(d)** by 15 to get v_{15}.

$$= \frac{61}{16}(2^m) + 6\left[\frac{125}{64}(2^m) + \left(\frac{125}{64}(2^m) - 275\right)\right]$$
$$= \frac{61}{16}(2^m) + 6\left[\frac{125}{32}(2^m) - 275\right]$$
$$= \frac{109}{4}(2^m) - 1650 \quad \text{where } s = \frac{109}{4},\ t = -1650$$

(f) Based on revised operating procedure, amount of data processed after a certain number of days will become negative due to AP with negative common difference. Since it is not possible to process negative amount of data, it is not meaningful to compute such data in the long run.

13

(a) Sum of the all the terms after the nth term

$= S_\infty - S_n$

$= \dfrac{a}{1-r} \quad \dfrac{a\left(1-r^n\right)}{1-r}$

$= \dfrac{a}{1-r}\left[1-\left(1-r^n\right)\right]$

$= \dfrac{ar^n}{1-r}$

Given $S_\infty - S_n = 2u_n$, therefore

$\dfrac{ar^n}{1-r} = 2ar^{n-1}$

$\dfrac{r}{1-r} = 2$

$r = 2(1-r)$

$r = \dfrac{2}{3}$

Hence $S_\infty = \dfrac{a}{1-r} = \dfrac{a}{1-\dfrac{2}{3}} = 3a$ (Shown)

(b)(i) Total number of integers in the first (r–1)th brackets is

$1+2+3+...+(r-1) = \dfrac{r-1}{2}\left(1+(r-1)\right) = \dfrac{r\left(r-1\right)}{2}$

Hence, first integer in the rth bracket $= \dfrac{r\left(r-1\right)}{2} + 1 = \dfrac{r^2-r+2}{2}$

Last integer in the rth bracket

$= \dfrac{r^2-r+2}{2} + (r-1)$

$= \dfrac{r^2-r+2+2r-2}{2}$

$= \dfrac{r^2+r}{2}$

<u>Alternative method</u>:

Last integer in the rth bracket

= First integer in the (r+1)th bracket minus 1

$= \left[\dfrac{\left(r+1\right)\left(r\right)}{2} + 1\right] - 1 = \dfrac{r^2+r}{2}$

(b)(ii) There are r integers in the rth bracket.

First integer in the rth bracket $= \dfrac{r^2-r+2}{2}$

Last integer in the rth bracket $= \dfrac{r^2+r}{2}$

Sum of all the integers in the rth bracket

$= \dfrac{r}{2}\left(\dfrac{r^2-r+2}{2} + \dfrac{r^2+r}{2}\right) = \dfrac{r}{2}\left(\dfrac{2r^2+2}{2}\right)$

$= \dfrac{1}{2}r\left(1+r^2\right)$ (Shown)

14

(a) 1, 3, 5, 7, ...

Number of terms in each bracket follows an AP with first term 1 and common difference 2.

$\therefore$ Number of integers in the first n sets

$=\frac{n}{2}\left[2(1)+(n-1)2\right]=n^2$

(b) Last integer in the nth set is the $\left(n^2\right)$th term of the AP

1, 4, 7, 10, 13, 16, ... which has first term 1 and common difference 3.

Last integer of the nth set $=1+\left(n^2-1\right)3=3n^2-2$

From GC,

n	$3n^2-2$
25	1873
26	2026

$\therefore\ k=26$

OR:

Given that 2023 occurs in the kth set,

first term in the kth set $\le 2023 \le$ last term in the kth set

$$\left[3(k-1)^2-2\right]+3\le 2023\le 3k^2-2$$

$$(k-1)^2\le\frac{2022}{3}\quad\text{and}\quad k^2\ge\frac{2025}{3}$$

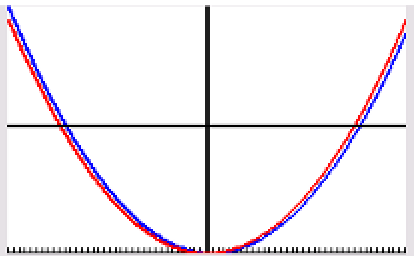
$-24.961\le k\le 26.961$ and $k\le -25.980$ or $k\ge 25.980$

$\therefore 25.980\le k\le 26.961$

Since $k\in\mathbb{Z}^+$, $k=26$

OR:

$\left[3(k-1)^2-2\right]+3\le 2023\le 3k^2-2$



From GC,

$-24.961\le k\le 26.961$ and $k\le -25.980$ or $k\ge 25.980$

$\therefore 25.980\le k\le 26.961$

Since $k\in\mathbb{Z}^+$, $k=26$

(c) Required sum

$= \text{Sum of first } \left(10^2\right)\text{th terms} - \text{Sum of first } \left(4^2\right)\text{th terms}$

$= \dfrac{10^2}{2}\left[2(1)+\left(10^2-1\right)3\right] - \dfrac{4^2}{2}\left[2(1)+\left(4^2-1\right)3\right]$

$= 14\ 574$

OR:

Last term in the 10th set $= 3(10)^2 - 2 = 298$

Last term in the 4th set $= 3(4)^2 - 2 = 46$

First term in the 5th set $= 46 + 3 = 49$

To find the sum of the AP : $49, 52, 55, \ldots, 298$ with first term 49 and common difference 3:

$298 = 49 + \left(m-1\right)3$

$m = 84$

$\therefore$ Required sum $= \dfrac{84}{2}\left(49+298\right) = 14\ 574$

OR using GC:

Since the 1st integer in the 5th set is the (4^2+1)th term in the AP and the last integer in the 10th set is the 10^2th term,

required sum $= \displaystyle\sum_{r=17}^{100} 1 + 3\left(r-1\right) = 14574$

15

(i) Day 3: 8 $= 8^1$ new squares

Day 4: 64 $= 8^2$ new squares

Day N: 8^{N-2} new squares

(ii)

Day 2: $\left(\frac{1}{3}\right)x$

Day 3: $\frac{1}{3}\left(\frac{1}{3}\right)x = \left(\frac{1}{3}\right)^2 x$

Day 4: $\frac{1}{3}\left(\frac{1}{3}\right)^2 x = \left(\frac{1}{3}\right)^3 x$

Day N: $\left(\frac{1}{3}\right)^{N-1} x$

(iii) Total perimeter of new squares drawn on Day N is

$$(8)^{N-2}\left(\frac{1}{3}\right)^{N-1} x(4) = (8)^{N-1}\left(\frac{1}{8}\right)\left(\frac{1}{3}\right)^{N-1} x(4) = \frac{4x}{8}\left(\frac{8}{3}\right)^{N-1} = \frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$$

(iv) Total perimeter of all the squares the artist draws by the end of the Nth day is

$$4x + \frac{x}{2}\left(\frac{8}{3}\right)^{2-1} + \frac{x}{2}\left(\frac{8}{3}\right)^{3-1} + \frac{x}{2}\left(\frac{8}{3}\right)^{4-1} + \ldots + \frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$$

$$= 4x + \frac{x}{2}\left(\frac{8}{3}\right)^{1} + \frac{x}{2}\left(\frac{8}{3}\right)^{2} + \frac{x}{2}\left(\frac{8}{3}\right)^{3} + \ldots + \frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$$

$$= 4x + \frac{\frac{x}{2}\left(\frac{8}{3}\right)^{1}\left(\left(\frac{8}{3}\right)^{N-1} - 1\right)}{\left(\left(\frac{8}{3}\right) - 1\right)}$$

$$= 4x + \frac{\frac{4x}{3}\left(\left(\frac{8}{3}\right)^{N-1} - 1\right)}{\frac{5}{3}}$$

$$= 4x + \frac{4x}{5}\left(\left(\frac{8}{3}\right)^{N-1} - 1\right)$$

$$= \frac{4x}{5}\left(\frac{8}{3}\right)^{N-1} + \frac{16}{3}x$$

(v) Total number of squares drawn by the end of Day 6 is

$1+8^0+8^1+8^2+8^3+8^4=4682$

The director will pay for the 1st, 4th, 7th, 10th, … .

$T_n \le 4682$

$1+(n-1)(3)\le 4682$

$n \le 1561\frac{1}{3}$

Thus there is a total of 1561 squares that the director will be paying.

He will be paying \$10 for the 1st square, \$13 for the 4th square, \$16 for the 7th square, and so on.

$\frac{1561}{2}\left[2(10)+(1561-1)3\right]$

$=\frac{1561}{2}\left[20+1560\times 3\right]$

$=3668350$

The amount of money the director will personally pay is \$3,668,350.