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$$\begin{aligned}
 \text{(a)} \quad u_n = S_n - S_{n-1} &= 18 - \frac{2^{n+1}}{3^{n-2}} - \left( 18 - \frac{2^n}{3^{n-3}} \right) \\
 &= \frac{2^n}{3^{n-3}} - \frac{2^{n+1}}{3^{n-2}} \\
 &= \frac{(3)2^n}{(3)3^{n-3}} - \frac{2^{n+1}}{3^{n-2}} \\
 &= \frac{2^n(3-2)}{3^{n-2}} = \frac{2^n}{3^{n-2}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{u_n}{u_{n-1}} &= \frac{\frac{2^n}{3^{n-2}}}{\frac{2^{n-1}}{3^{n-3}}} \\
 &= \frac{2^n}{3^{n-2}} \times \frac{3^{n-3}}{2^{n-1}} \\
 &= \frac{2^{n-(n-1)}}{3^{n-2-(n-3)}} \\
 &= \frac{2}{3}
 \end{aligned}$$

Since  $\frac{u_n}{u_{n-1}}$  is a constant independent of  $n$ , the sequence is a geometric progression.

$$\text{(b)} \quad \text{As } n \rightarrow \infty, S_n = 18 - \frac{2^{n+1}}{3^{n-2}} = 18 - \frac{2(2^n)}{3^{-2}(3^n)} = 18 - 18\left(\frac{2}{3}\right)^n \rightarrow 18$$

(Or using G.P. sum to infinity formula  $S_\infty = \frac{6}{1 - \frac{2}{3}} = 18$ )

Let the common difference be  $d$ .

$$\frac{9}{2}[2(-4) + 8d] = 18$$

$$-8 + 8d = 4$$

$$d = \frac{3}{2}$$

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$$S_n = an + \frac{b}{n+1} + c$$

$$S_1: a + \frac{b}{2} + c = \frac{5}{2} \text{ ----- (1)}$$

$$u_2 = S_2 - \frac{5}{2}: 2a + \frac{b}{3} + c - \frac{5}{2} = \frac{13}{6}$$

$$2a + \frac{b}{3} + c = \frac{14}{3} \text{ ----- (2)}$$

$$S_5: 5a + \frac{b}{6} + c = \frac{65}{6} \text{ ----- (3)}$$

From GC,  $a = 2, b = -1, c = 1$

$$\text{Hence, } S_n = 2n - \frac{1}{n+1} + 1$$

$$u_n = S_n - S_{n-1}$$

$$= \left(2n - \frac{1}{n+1} + 1\right) - \left(2(n-1) - \frac{1}{n-1+1} + 1\right)$$

$$= 2n - \frac{1}{n+1} + 1 - 2n + 2 + \frac{1}{n} - 1$$

$$= 2 - \left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$= 2 - \left(\frac{n - (n+1)}{n(n+1)}\right) = 2 + \frac{1}{n(n+1)}$$

As  $n$  gets larger,  $\frac{1}{n(n+1)}$  gets smaller. So  $u_n > u_{n+1}$ .

As  $n \rightarrow \infty, \frac{1}{n(n+1)} \rightarrow 0, u_n \rightarrow 2$ .

The sequence is decreasing ( $u_n > u_{n+1}$ ) and converges to 2.

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(a) **Method ①:**

The  $n$ th term,  $u_n$ , is always one degree less than  $S_n$  since  $u_n = S_n - S_{n-1}$ .

If  $S_n$  is quadratic,  $u_n$  would be linear but it is not since there is no common difference between consecutive terms.

**Method ②: Proof by Contradiction**

Suppose  $S_n = an^2 + bn + c$

$$S_n = an^2 + bn + c$$

$$a + b + c = -4$$

$$4a + 2b + c = -4 - 2 = -6$$

$$9a + 3b + c = -6 + 12 = 6$$

$$16a + 4b + c = 6 + 38 = 44$$

Using G.C., no solution found.

Hence  $S_n$  cannot be a quadratic polynomial.

(b)  $S_n = an^3 + bn^2 + cn + d$

$$a + b + c + d = -4$$

$$8a + 4b + 2c + d = -4 - 2 = -6$$

$$27a + 9b + 3c + d = -6 + 12 = 6$$

$$64a + 16b + 4c + d = 6 + 38 = 44$$

Using G.C.,

$$a = 2, b = -5, c = -1, d = 0$$

$$S_n = 2n^3 - 5n^2 - n$$

(c)  $u_n = S_n - S_{n-1}$

$$= 2n^3 - 5n^2 - n - [2(n-1)^3 - 5(n-1)^2 - (n-1)]$$

$$= 2n^3 - 5n^2 - n - [2(n^3 - 3n^2 + 3n - 1) - 5(n^2 - 2n + 1) - n + 1]$$

$$= 2n^3 - 5n^2 - n - (2n^3 - 11n^2 + 15n - 6)$$

$$= 6n^2 - 16n + 6$$

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(a)  $u_{n+1} = 0.1u_n + k$   
 $u_2 = 0.1u_1 + k = 0.1(-3) + k = k - 0.3$   
 $u_3 = 0.1u_2 + k = 0.1(k - 0.3) + k = 1.1k - 0.03$

(b) We are given  $u_n = \beta + \alpha \left(\frac{1}{10}\right)^n$

When  $n = 1$ ,

$$-3 = \beta + \alpha \left(\frac{1}{10}\right)^1$$

$$-3 = \beta + 0.1\alpha \quad \dots(1)$$

When  $n = 2$ ,

$$k - 0.3 = \beta + \alpha \left(\frac{1}{10}\right)^2$$

$$k - 0.3 = \beta + 0.01\alpha \quad \dots(2)$$

(1) - (2),

$$-3 - (k - 0.3) = (\beta + 0.1\alpha) - (\beta + 0.01\alpha)$$

$$-2.7 - k = 0.09\alpha$$

$$\alpha = \frac{100}{9}(-2.7 - k)$$

$$= -\frac{100}{9}k - 30$$

Sub into (1),

$$-3 = \beta + \frac{1}{10} \left( -\frac{100}{9}k - 30 \right)$$

$$-3 = \beta - \frac{10}{9}k - 3$$

$$\beta = \frac{10}{9}k$$

(c)  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left( \frac{10}{9}k + \alpha \left(\frac{1}{10}\right)^n \right) = \frac{10}{9}k$

(d) When  $k = 9$ ,  $\alpha = -\frac{100}{9}(9) - 30 = -130$ .

$$u_n = 10 - 130 \left(\frac{1}{10}\right)^n$$

$$\sum_{i=1}^n u_i = \sum_{i=1}^n \left[ 10 - 130 \left(\frac{1}{10}\right)^i \right]$$

$$= \sum_{i=1}^n 10 - 130 \sum_{i=1}^n \left(\frac{1}{10}\right)^i$$

$$= 10n - (130) \frac{\frac{1}{10} \left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}$$

$$= 10n - \frac{130}{9} \left(1 - \frac{1}{10^n}\right)$$

$$= 10n - \frac{130}{9} + \frac{130}{9(10^n)}$$

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(a) Let first term be  $u_1 = a$  and common difference  $d$ .

$$\text{Middle term } u_{\frac{n+1}{2}} = a + \left( \frac{n+1}{2} - 1 \right) d$$

$$\text{Last term } u_n = a + (n-1)d$$

$$\text{Given } a + a + \left( \frac{n+1}{2} - 1 \right) d = a + (n-1)d$$

$$a + \frac{nd}{2} + \frac{d}{2} - d = nd - d$$

$$\frac{nd}{2} = a + \frac{d}{2}$$

$$n = \frac{2a + d}{d}$$

$$n = \frac{2a}{d} + 1 \quad \text{OR} \quad d = \frac{2a}{n-1}$$

Sub  $n = \frac{2a}{d} + 1$  into the middle term  $u_{\frac{n+1}{2}} = a + \left( \frac{n+1}{2} - 1 \right) d$ :

$$u_{\frac{n+1}{2}} = a + \left( \frac{\frac{2a}{d} + 1 + 1}{2} - 1 \right) d$$

$$= a + \left( \frac{\frac{2a}{d} + 2}{2} - 1 \right) d$$

$$= a + \left( \frac{a}{d} + 1 - 1 \right) d$$

$$= a + \left( \frac{a}{d} \right) d$$

$$= 2a$$

**OR Alternatively**

Sub  $d = \frac{2a}{n-1}$  into the middle term  $u_{\frac{n+1}{2}} = a + \left( \frac{n+1}{2} - 1 \right) d$ :

$$u_{\frac{n+1}{2}} = a + \left( \frac{n+1}{2} - 1 \right) \left( \frac{2a}{n-1} \right)$$

$$= a + \left( \frac{n-1}{2} \right) \left( \frac{2a}{n-1} \right)$$

$$= 2a$$

**(b)(i)** Given  $T_1 = 3p^3$ ,  $r = \frac{p}{q^2}$

And

$$m = 3p^3 \left( \frac{p}{q^2} \right)^{n-1}$$

$$m = 3p^3 \left( \frac{p^{n-1}}{q^{2(n-1)}} \right)$$

$$m = 3 \left( \frac{p^{n+2}}{q^{2n-2}} \right)$$

$$\ln m = \ln 3 + \ln p^{n+2} - \ln q^{2n-2}$$

$$\ln m = \ln 3 + (n+2) \ln p - (2n-2) \ln q$$

$$A = 2, B = 2$$

**(b)(ii)** Since it is a convergent GP, then:  $|r| < 1$ .

$$-1 < \frac{p}{q^2} < 1$$

$$q = 2p$$

$$-1 < \frac{p}{(2p)^2} < 1$$

$$-1 < \frac{1}{4p} < 1$$

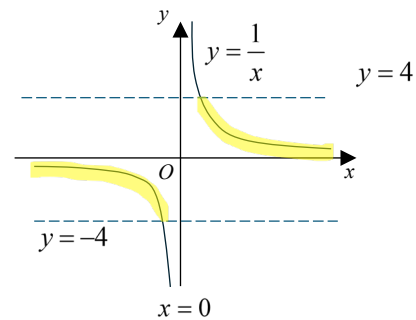
$$-4 < \frac{1}{p} < 4$$

$$p < -\frac{1}{4}$$

OR

$$p > \frac{1}{4}$$

(Rejected,  
since all terms  
are positive.)



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(a)(i) (A)  $c = 5$

The sequence initially **decreases** and subsequently **alternates**, and **converges to 2**.

$n$	$u_n$			
10	1.9941			
11	2.0029			
12	1.9985			
13	2.0007			
14	1.9996			
15	2.0002			
16	1.9999			
17	2			
18	2			
19	2			
20	2			

$n=10$

(B)  $c = 2$

It is a **constant** with a value of **2**.

$n$	$u_n$			
1	2			
2	2			
3	2			
4	2			
5	2			
6	2			
7	2			
8	2			
9	2			
10	2			
11	2			

$n=1$

(a)(ii)  $u_1 = c$

$$u_2 = 3 - 0.5u_1 = 3 - 0.5c$$

$$u_3 = 3 - 0.5u_2 = 3 - 0.5(3 - 0.5c) = 1.5 + 0.25c$$

$$\text{Given } 2u_3 = -5u_2$$

$$2(1.5 + 0.25c) = -5(3 - 0.5c)$$

$$3 + 0.5c = -15 + 2.5c$$

$$2c = 18$$

$$c = 9$$

(b)(i) Given  $v_3 + 1 = 2u_2 - u_1$ ,

$$2v_1 + v_2 - 1 + 1 = 2u_2 - u_1,$$

$$2p + 2 = 2(3 - 0.5c) - c$$

$$2p + 2 = 6 - 2c$$

$$c + p = 2$$

(b)(ii) Given  $v_1 = p, v_2 = 2$ ,

$$v_3 = 2v_1 + v_2 - 1 = 2p + 2 - 1 = 2p + 1$$

$$v_4 = 2v_2 + v_3 - 1 = 2(2) + (2p + 1) - 1 = 2p + 4$$

$$v_5 = 2v_3 + v_4 - 1 = 2(2p + 1) + (2p + 4) - 1 = 6p + 5$$

$$\therefore 6p + 5 = 77$$

$$\therefore p = 12$$

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(i) When  $n = 0$ ,

$$u_1 - u_0 = -\frac{4}{3}\left(\frac{1}{3}\right)^0 + Q\left(\frac{2}{3}\right)^0$$

$$\frac{4}{3} - 3 = -\frac{4}{3}\left(\frac{1}{3}\right)^0 + Q\left(\frac{2}{3}\right)^0$$

$$Q = \frac{8}{3} - 3 = -\frac{1}{3} \text{ and}$$

$$u_2 - u_1 = -\frac{4}{3}\left(\frac{1}{3}\right)^1 - \frac{1}{3}\left(\frac{2}{3}\right)^1$$

$$u_2 = -\frac{4}{3}\left(\frac{1}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\right) + \frac{4}{3} = \frac{2}{3}$$

(ii)

$$|u_{n+1} - u_n| = \left| -\frac{4}{3}\left(\frac{1}{3}\right)^n - \frac{1}{3}\left(\frac{2}{3}\right)^n \right| = \left| -\frac{1}{3} \left[ 4\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n \right] \right|$$

$$|u_{n+1} - u_n| = \frac{1}{3} \left| 4\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n \right| \leq \frac{1}{3} \left| 4\left(\frac{2}{3}\right)^n + \left(\frac{2}{3}\right)^n \right|$$

$$|u_{n+1} - u_n| \leq \frac{5}{3} \left(\frac{2}{3}\right)^n$$

Thus  $|u_{n+1} - u_n| \leq \varepsilon$

when  $\frac{5}{3} \left(\frac{2}{3}\right)^n \leq \varepsilon$

$$\left(\frac{2}{3}\right)^n \leq \frac{3\varepsilon}{5}$$

$$n \geq \frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)}$$

$$n_0 \text{ can be } \left\lceil \frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)} \right\rceil \text{ or ceiling of } \frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)}.$$

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$$\text{When } \varepsilon = 0.001, n \geq \frac{\ln\left(\frac{3\varepsilon}{5}\right)}{\ln\left(\frac{2}{3}\right)} = 18.29$$

$n_0$  can be 19.